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Removal of Goldstein's singularity at separation, in flow past obstacles in wall layers

By F. T. SMITH
Mathematics Department, Imperial College, London SW7 2BZ
AND P. G. DANIELS
Mathematics Department, City University, St John Street, London EC1V 4PB

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It is shown that, in the flow of a viscous wall layer past a relatively steep obstacle at the wall, the Goldstein (1948) singularity generated in the classical boundary-layer approach to separation *is* removable in a physically sensible fashion. The removal is effected by means of a sequence of local double structures, the last of which arises just beyond separation owing to the occurrence of a further singularity which is also removable and describes the necessary complete breakaway of the viscous layer from the wall. The novel forms of the local pressure–displacement relations are the key elements allowing the solution to retain physical reality throughout. Beyond the breakaway the reattachment process takes place only at a relatively large distance downstream, before the motion returns to its original uniform shear form. The present flow configuration, the first we know of where Goldstein's singularity proves to be removable, has important applications in both internal and external flows at high Reynolds numbers and these are also discussed.

1. Introduction

The Goldstein (1948) singularity at separation and the three fundamental issues it raises lie at the very heart of the theory of high-Reynolds-number flows.

His singularity is based on the apparently self-consistent local argument that in the approach of a boundary layer towards a separation point the scaled skin friction can tend to zero in a singular fashion, being proportional to the square root of the distance upstream of separation, according to a classical theory where the necessarily adverse pressure gradient driving the boundary layer towards separation is known in advance. The first issue provoked by Goldstein's argument concerns the question of whether or not the singularity does actually occur in a classical boundary-layer solution at the onset of separation. However this issue has largely been settled by accurate numerical solutions of classical boundary layers under prescribed adverse pressure gradients (see the review by Brown & Stewartson 1969) which point firmly to the occurrence of the Goldstein singularity in almost all circumstances of interest. The next issue then is the matter of whether the singularity is welcome or not. In other words, can it be meaningfully removed, in the sense of being smoothed out in a physically realistic manner by means of a new flow structure more closely surrounding the implied separation point, or does its appearance instead herald the collapse of the classical theory? The only serious and conclusive attempt to respond to this question was made in an important paper by Stewartson (1970*a*). From an analysis addressing the separation of a classical boundary layer driven by a subsonic or supersonic decelerating mainstream he found strong evidence in favour of the view that the Goldstein singularity cannot be removed in a physically sensible way. In the supersonic context, for instance, Stewartson showed that a removal of the singularity almost certainly leads to an unrealistic and worse singularity in the local flow solution. There is little doubt that a similar conclusion also holds in many other contexts including transonic or hypersonic mainstreams and many internal flows. So it seems that the strategy of classical boundary-layer theory then leads to a catastrophe, and can be regarded as a failure, when separation is present. This of course raises the final issue: is there an alternative and successful strategy? The answer, a firm 'Yes', is provided by the theory of viscous–inviscid interactions whereby a well-attached boundary layer is able to separate fairly abruptly owing to the action of a small pressure rise, but large adverse pressure gradient, spread over a relatively small streamwise length scale (in contrast with the classical supposition of a more gradual pressure rise). This alternative strategy started with the triple-deck structure of Stewartson & Williams (1969), Stewartson (1970*b*) and Messiter (1970), has subsequently led to complete accounts of separation in external and internal flow situations and has been summarized by Stewartson (1974*a*, 1980), Messiter (1979) and Smith (1979*c*).

It should not be inferred from the above that in *all* flow situations, once the Goldstein singularity is encountered at a separation, the theory leading thereto is automatically a failure, however. Neither should it be concluded that there is always a successful strategy alternative to the classical boundary-layer one. For the present paper presents a flow situation in which there seems to be no possible approach other than a classical boundary-layer treatment upstream of separation and in which the resulting Goldstein singularity at separation *does* prove to be removable and in an eminently realistic fashion. The case of interest to us is not especially bizarre or contrived, incidentally. It arises in the fluid motion past an isolated obstacle in a wall layer, has a wide range of application in both external and internal flows (Smith 1973, 1976*a*, *b*; Smith *et al.* 1981) and it concerns the solution of the boundary-layer equations subject to a given smooth displacement of the uniform shear flow holding far from the wall.

The point is that the issue of whether the Goldstein singularity is removable in a physically sensible manner or not is usually a rather delicate matter, as is the issue of the existence of an alternative strategy; in particular both issues hinge on the properties of the interaction set up locally between the induced pressure variation and the boundary-layer displacement in the neighbourhood of separation. Now for the flow situations mentioned originally where the boundary layer is driven by an external mainstream, and in certain internal flows also, it turns out that the local pressure–displacement interaction is such that increases in the pressure and displacement are mutually reinforcing (at least in supersonic flow: subsonic flow is less easy to interpret on this point). This feature is then the cause both of the unrealistic flow behaviour found when a removal of the Goldstein singularity is attempted and of the existence of the alternative strategy of viscous–inviscid interactions which has proved successful in describing both supersonic and subsonic separation (Stewartson & Williams 1969; Sychev 1972; Messiter 1976; Smith 1977*a*). So in such flow situations there seems little room for doubt about the nature of separation. But not all pressure–displacement interactions are mutually reinforcing. Indeed, the flow situation that concerns us happens to provide an important example where the interaction is not mutually reinforcing; as a result the Goldstein singularity can be removed without any loss of physical reality.

As well as having certain repercussions for the Goldstein singularity and hence for high-Reynolds-number flow theory in general, the present study is believed to yield also certain quite significant properties in the specific contexts of flow over obstacles in external boundary layers and of internal flow through constricted tubes. In the former context, which obviously has strong possible applications in aerodynamics and atmospheric dynamics, the study enables the extension to be made, beyond the theory of Smith (1972, 1973) and Smith *et al.* (1981), to the understanding of more grossly separated external supersonic and subsonic motions. In the context of the internal motions, where the possible applications are to physiological flows and machinery dynamics, the study complements those of Smith (1976*a*, *b*, 1978, 1979*a*), particularly with regard to the separation from the constriction and the resultant long recirculating eddy or eddies. In both contexts the present work provides still further firm evidence in favour of the extended free streamline description as the correct limiting inviscid form for grossly separated flows. That description is almost certainly correct in internal flows (Smith 1979*a*; Smith & Duck 1980), is quite possibly correct in external flows also (Sychev 1967; Messiter 1975; Smith 1979*b*), and is found to be correct beyond the viscous separation in the situation which concerns us here.

The plan and main findings of this paper are as follows. Section 2 introduces the steady laminar two-dimensional flow problem, which is expressed in terms of the boundary-layer equations for the velocity u in the x direction, the stream function ψ and the pressure p. This is in a non-dimensional form as set down by Smith (1976*a*, *b*) and Smith *et al.* (1981) and its full relation to external and internal flows is specified later in § 6. The solution depends only on a single parameter h which effectively measures the lateral extent of the obstacle relative to the viscous wall-layer thickness. Thus the surface of the obstacle is given by $\tilde{y} = hF(x)$, where the given function $F(x)$ is independent of h and $\tilde{y}$ denotes the scaled lateral co-ordinate measured from the undisturbed wall $\tilde{y} = 0$. It is required that the original shear flow $u = \tilde{y}$ far upstream should remain undisplaced far from the obstacle so that $u - \tilde{y} \to 0$ as $\tilde{y} \to \infty$. The task then is to find the flow structure and properties holding when h is large. A classical boundary-layer strategy seems inevitable and it leads perforce to the Goldstein singularity at the approach to separation (§ 2). The clue to the first new small length scale necessary closer to separation is provided by the properties of the small corrections (§ 2) to the classical strategy and the new length scale, studied in § 3.1, then merely causes a slight shift of origin in the singularity. Hence a shorter length scale comes into operation even nearer separation. It is on this second length scale that Goldstein's singularity proves to be sensibly removable and regular separation occurs (§ 3.2). Beyond the separation a worse singularity is then enountered but it retains physical sense, with the reversed flow velocities and the wall-layer displacement being continually enhanced during the process, due to the nature of the local pressure–displacement laws (see also above). Therefore we move on to consider removal of the worse second singularity in § 4. Here a still shorter length scale controls the flow behaviour, involving a nonlinear response very close to the wall, the gradual breakaway from the wall of the entire original wall layer and the eventual fading out of the adverse pressure gradient downstream as the breakaway process is completed. The whole separation and breakaway process takes place through a sequence of double structures. Beyond the breakaway (§ 5) extended free-streamline theory governs the grossly separated flow structure for a relatively large distance downstream until reattachment is forced by the thickening of the detached shear layer. The reattachment is sufficiently influenced by viscous diffusion that no significant extra effects are provoked upstream of the reattachment stage, while downstream the flow ultimately retrieves the uniform shear form. The final discussion in § 6 includes comments not only on our particular flow problem, and the realistic removal of Goldstein's singularity, but on the broader implications also.

2. The arrival at Goldstein's singularity at separation

The flow problem of concern to this study arises in high-Reynolds-number motion, most significantly in internal flow through a tube (Smith 1976*a, b*, 1978, 1979*a*) but also in certain external flows whether supersonic or subsonic (Smith 1973; Smith *et al.* 1981). It is to solve the incompressible boundary-layer equations

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \quad u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -p'(x)+\frac{\partial^2 u}{\partial y^2}, \tag{2.1a}$$

with the boundary conditions

$$u = \psi = 0 \quad \text{at} \quad y = 0, \tag{2.1b}$$

$$v, p \to 0, \quad u \to y \quad \text{as} \quad x \to -\infty, \tag{2.1c}$$

$$u \sim y + hF(x) \quad \text{as} \quad y \to \infty. \tag{2.1d}$$

These describe the effects of an obstacle at the wall on the otherwise uniform shear flow in a wall layer. The Prandtl transformation $y = \tilde{y} - hF(x)$ has been applied already for convenience, so that the outer boundary condition (2.1*d*) reflects the required absence of any displacement of the original motion $u = \tilde{y}$, $v = p = 0$ as mentioned in the introduction. The scaled obstacle shape $F(x)$, independent of h, is given and is assumed to be continuous and of continuous slope also although the latter condition could be relaxed at certain stations, at the start or finish of an obstacle of finite length for instance, without disturbing the ensuing investigation. Again, we will suppose that $F(\pm\infty) = 0$, giving an undisturbed wall far upstream and downstream, that the obstacle shape achieves a unique maximum $F_{\max}$, at $x = x_{\max}$ say ($F(x_{\max}) = F_{\max}$), and that the slope is monotonic on either side of the maximum, $F'(x) \gtreqless 0$ for $x \lesseqgtr x_{\max}$, giving a hump rather than a dent on the wall. Relaxation of certain of these conditions can lead to some extra interesting features but they are of relatively peripheral concern. More important is the height parameter h which effectively gauges the unscaled obstacle height relative to the unscaled wall-layer thickness in the original expansions, in terms of the Reynolds number, used to derive (2.1*a–d*) from the Navier–Stokes equations (see references above). In such expansions h is regarded as $O(1)$ of course, but, once the closed problem (2.1*a–d*) is encountered, free of Reynolds-number dependence, all its solutions for $0 < h < \infty$ are equally relevant to the original formal treatment of the Navier–Stokes equations.

In (2.1*a–d*) the obstacle height $hF(x)$ plays the role of a given negative displacement,

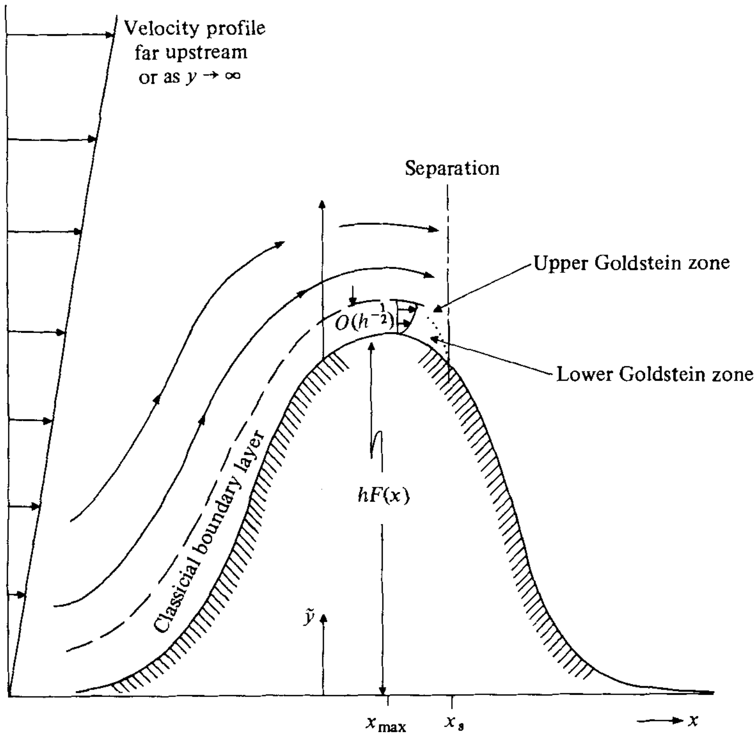

FIGURE 1. The general features of the wall layer flow (§2) past the obstacle $\tilde{y} = hF(x)$ when h is large, up to the onset of separation ($x \to x_s -$).

while the pressure $p(x)$ is an unknown function of x. Although linearized solutions for small h were analysed by Smith (1976*b*) and Smith *et al.* (1981), the numerical solutions of the nonlinear stage $h = O(1)$ presented by Smith (1976*a*) are of more relevance. The latter solutions include some which exhibit separation, at a station where the skin friction $\tau(x) = \partial u(x, 0)/\partial y$ passes through zero, followed by reversed flow. The separations there are almost certainly regular phenomena, in view of the prescribed smoothness of the displacement in (2.1*d*) (see Catherall & Mangler 1966; Brown & Stewartson 1969; and below), and the extent of the reversed flow region downstream grows with h for a given shape $F(x)$, as one might expect physically. This raises the question, to be addressed henceforth, of what happens to the solution of (2.1*a*–*d*) as $h \to \infty$?

For $h \gg 1$ the form of the boundary conditions (2.1*c*, *d*) suggests trying the orderings $y = O(h)$, $u = O(h)$ first, with x of $O(1)$ because of the shape dependence $F(x)$. These orderings are also in line with the alternative outer constraint

$$\psi \sim \tfrac{1}{2}(y + hF(x))^2 + p(x) \quad \text{as} \quad y \to \infty, \tag{2.1e}$$

obtained from integration of (2.1*d*) and substitution into (2.1*a*, *c*). For (2.1*e*) suggests $\psi = O(h^2)$, $p = O(h^2)$ when $y = O(h)$. Then the governing equations (2.1*a*) imply that $y = O(h)$ defines only an outer inviscid region, viscous effects being confined to a thinner layer wherein y is smaller, of $O(h^{-\frac{1}{2}})$, and so giving a relative effect of order $h^{-\frac{3}{2}}$ in the outer flow (figure 1). In fact the outer flow is found to involve the direct continuation of the boundary condition (2.1 *e*). Thus with the suggested expansion

$$p(x) = h^2 p_0(x) + h^{\frac{1}{2}} p_1(x) + \dots \tag{2.2}$$

for the pressure we propose that the expressions

$$\psi = h^2[\tfrac{1}{2}(\bar{y} + F(x))^2 + p_0(x)] + h^{\frac{1}{2}} p_1(x) + \dots, \tag{2.3a}$$

$$u = h(\bar{y} + F(x)) + O(\exp) \tag{2.3b}$$

describe the outer flow for $\bar{y} > 0$, where $y = h\bar{y}$. From (2.1 *e*) the forms (2.2), (2.3 *a*, *b*) obviously satisfy the whole problem (2.1 *a*–*d*) except for the no-slip condition (2.1 *b*), which, however, indicates setting the typical inviscid constraint of tangential flow as the wall is approached ($\bar{y} \to 0+$). From the $O(h^2)$ term of (2.3 *a*) therefore we can fix the leading-order pressure force as

$$p_0(x) = -\tfrac{1}{2}F^2(x), \tag{2.4}$$

yielding a remarkably simple pressure-shape law (Smith 1978).

The thin viscous layer, with its $O(1)$ co-ordinate Y defined by $y = h^{-\frac{1}{2}} Y$, then has the form

$$\psi = h^{\frac{1}{2}} \psi_0(x, Y) + h^{-1} \psi_1(x, Y) + \dots, \tag{2.5a}$$

$$u = h u_0(x, Y) + h^{-\frac{1}{2}} u_1(x, Y) + \dots, \tag{2.5b}$$

because of (2.3 *a*, *b*), (2.4) as $\bar{y} \to 0+$. Substitution of (2.5 *a*, *b*) with (2.2) into (2.1 *a*) leaves the classical boundary-layer equations

$$u_0 = \frac{\partial \psi_0}{\partial Y}, \quad u_0 \frac{\partial u_0}{\partial x} - \frac{\partial \psi_0}{\partial x}\frac{\partial u_0}{\partial Y} = F(x)\,F'(x) + \frac{\partial^2 u_0}{\partial Y^2} \tag{2.6a}$$

governing u_0, ψ_0, where use is made of the known pressure gradient from (2.4). The expected boundary conditions appropriate to the viscous layer are

$$u_0 = \psi_0 = 0 \quad \text{at} \quad Y = 0, \tag{2.6b}$$

$$u_0 \to F(x) \quad \text{as} \quad Y \to \infty, \tag{2.6c}$$

$$u_0, \psi_0 \to 0 \quad \text{as} \quad x \to -\infty, \tag{2.6d}$$

from (2.1 *b*, *c*) and from joining the expansions (2.5 *a*, *b*) with those of (2.3 *a*, *b*). Hence to leading order the viscous layer here is no more than a classical boundary layer driven by a known pressure gradient, $-F(x)F'(x)$. Since $F(x) > 0$ the pressure gradient is favourable when the obstacle slope $F'(x)$ is positive, upstream of $x = x_{\max}$, and so the solution of (2.6 *a*–*d*) can be taken to exist up to $x = x_{\max}$ and somewhat beyond; an example is calculated by Smith (1978), albeit for an obstacle shape different from those we have in mind; see also figure 9 (*b*) below. The solution can be found by numerical means in general (as in figure 9 *b* below), advancing from the initial state of rest in (2.6 *d*), and in particular determines the viscous-layer displacement $\beta_0(x)$, defined by the behaviour

$$\psi_0 \sim F(x)\,[Y - \beta_0(x)] \quad \text{as} \quad Y \to \infty \tag{2.6e}$$

stemming from (2.6 *c*) with (2.6 *a*). Once $\beta_0(x)$ is known the outer solution (2.3 *a*) requires that

$$p_1(x) = -F(x)\,\beta_0(x), \tag{2.7}$$

which fixes the pressure perturbation; and in turn (2.7) provides the driving force for the perturbations within the viscous layer which are controlled by the boundary-layer disturbance equations

$$u_1 = \frac{\partial \psi_1}{\partial Y}, \quad u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x} - \frac{\partial \psi_1}{\partial x}\frac{\partial u_0}{\partial Y} - \frac{\partial \psi_0}{\partial x}\frac{\partial u_1}{\partial Y} = \{F(x)\,\beta_0(x)\}' + \frac{\partial^2 u_1}{\partial Y^2} \tag{2.8a}$$

in view of (2.5*a*, *b*) with (2.2) and (2.1*a*). The boundary conditions for (2.8*a*) are

$$u_1 = \psi_1 = 0 \quad \text{at} \quad Y = 0, \tag{2.8b}$$

$$\psi_1 \sim \tfrac{1}{2}Y^2 + O(1) \quad \text{as} \quad Y \to \infty, \tag{2.8c}$$

$$u_1 \to Y, \quad \psi_1 \to \tfrac{1}{2}Y^2 \quad \text{as} \quad x \to -\infty, \tag{2.8d}$$

for no slip at the wall, for matching with (2.3*a*) and for (2.1*c*) respectively. One could continue in the classical manner using the $O(1)$ term in (2.8*c*) to fix the next term of $O(h^{-1})$ in the pressure and so on.

In principle, therefore, the well-ordered schemes of classical boundary-layer theory provide the answer. *In fact*, however, they fail as usual in the general case where separation occurs, because of the approach of Goldstein's (1948) singularity in the solution of (2.6*a–d*) near separation, Admittedly, a separation might be avoided by having an obstacle shape with only mildly negative slopes beyond $x = x_{\max}$, thus producing in (2.6*a*) only a mildly adverse gradient, but we can regard that as just an interesting exception. In general the change in sign of the slope $F'(x)$ beyond the maximum obstruction will cause the scaled skin friction $\tau_0 = (\partial u_0/\partial Y)\,(x, 0)$ to tend to zero at a finite station $x = x_s > x_{\max}$, because of the adverse pressure gradient acting in $x > x_{\max}$. Then there is little doubt that because of its classical nature the solution of (2.6*a–d*) must acquire the Goldstein (1948) singular form as $x \to x_s-$. Certainly all reliable numerical studies of classical boundary layers approaching separation tend to verify that occurrence (Brown & Stewartson 1969), although a concrete analytical proof is lacking. We will take the occurrence of Goldstein's singularity at separation to be inevitable (see also figure 9*b* below). It has the following well-known form (figure 1).

As $x \to x_s-$ the flow of (2.6*a–d*) subdivides into two zones. First, for Y of $O(1)$, ψ_0 has the form

$$\psi_0 = \psi_{0s}(Y) + (x_s - x)^{\frac{1}{2}} \frac{2\alpha_0}{\mu} \psi'_{0s}(Y) + (x_s - x)^{\frac{3}{4}} \frac{2\alpha_1}{\mu} \psi'_{0s}(Y) + \dots \tag{2.9a}$$

from Goldstein (1948), where $\psi_{0s}(Y)$ is the stream-function profile at separation, satisfying

$$\psi_{0s}(Y) \sim \begin{cases} \tfrac{1}{6}\mu Y^3 - \tfrac{1}{60}\alpha_0^2\, Y^5 + \dots & \text{as } Y \to 0, \quad (2.9b) \\ F_s[Y - \beta_s] + o(1) & \text{as } Y \to \infty. \quad (2.9c) \end{cases}$$

Here (2.9*b*) reflects the vanishing of the skin friction $\psi''_{0s}(0)$ at $x = x_s$, while

$$F_s \equiv F(x_s) > 0 \quad \text{and} \quad F'_s \equiv F'(x_s) < 0$$

are finite, and

$$\mu = -F_s F'_s > 0 \tag{2.9d}$$

from (2.6*a*). Also (2.9*c*) is the outer boundary condition (2.6*e*) applied as $x \to x_s-$, with $\beta_0(x_s-) = \beta_s$ finite (see below), and the constants $\alpha_0, \alpha_1 (\propto \alpha_0^2)$ are unknown.

In the above the merging has already been anticipated with the second Goldstein zone, wherein Y is small and $O(x_s-x)^{\frac{1}{4}}$ and

$$\psi_0 = (x_s-x)^{\frac{3}{4}}\mu\eta^3/6+(x_s-x)\,\alpha_0\eta^2+(x_s-x)^{\frac{5}{4}}\,(\alpha_1\eta^2-\tfrac{1}{60}\alpha_0^2\eta^5)+\dots \tag{2.10}$$

with $\eta = Y(x_s-x)^{-\frac{1}{4}}$. The matched expansions (2.9a), (2.10) formally satisfy the boundary-layer equations (2.6a), with (2.6b, c, e), locally and give the singular behaviours of the displacement slope and skin friction,

$$\beta_0(x) \sim \beta_s-2\alpha_0\mu^{-1}(x_s-x)^{\frac{1}{2}}, \quad \tau_0(x) \sim 2\alpha_0(x_s-x)^{\frac{1}{2}}, \tag{2.11}$$

respectively. The constant α_0 remains undetermined, being dependent on the entire upstream flow for $x < x_s$ presumably, but it is believed to be non-zero, and so positive from (2.11), in general. Again, the coefficient of the square-root singularity in $\beta_0(x)$ in (2.11) is negative since μ must be positive for forward flow as $x \to x_s-$, from (2.9a, b), and the large positive slope implied for the displacement seems physically in line with an approach to separation. The suppression of separation until $x > x_{\max}$ is also verified, since F_s' (and hence $-p_0'(x)$) must be negative as noted in (2.9d).

The issue now is whether the Goldstein (1948) singular form of (2.9a)–(2.11) is removable, on a shorter length scale around separation, or whether instead the whole structure so far assumed is wrong and should be replaced. Stewartson (1970a) provided strong evidence supporting the view that the singularity is irremovable, at least in any physically meaningful way and in the context of external flow driven by a uniform stream. The resolution of the difficulty in those circumstances is now believed to be that the classical boundary-layer structure assumed globally does not yield a correct limiting description of the Navier–Stokes equations and that in its place the concept of the interactive triple-deck separation (Stewartson & Williams 1969; Sychev 1972; Smith 1977a), embodied in a quite different global structure which may well be of the extended free-streamline kind of Kirchhoff (1869), Sychev (1967), Messiter (1975) and Smith (1979b), must be introduced. The context is therefore somewhat different from ours, in mathematical as well as the obvious physical terms. For our governing equations are always those in (2.1a) and for any finite value of h their boundary conditions (2.1b–e) do not allow the Goldstein singularity to be present right at separation, if the idea of a prescribed smooth displacement forcing the separation to be regular is indeed correct, which we believe it is. Some backing for the idea is evident in (2.9a)–(2.11), incidentally, where regularity of the displacement would insist that α_0 be zero and similarly, at higher order, that α_1 be zero, and so on. In consequence we would expect the flow solution holding for $h \to \infty$ also to be regular right at separation, in contrast with the irregularities of (2.9a)–(2.11) just upstream. Again, there *seems* to be no alternative to the classical scheme of (2.2)–(2.8d) upstream of separation. Investigation shows that the alternative concept of an interactive process, whether leading to an upstream separation or not, tends to be ruled out by the imposed negative form of the effective displacement in (2.1d), although as usual one cannot claim to have covered all possible eventualities. In view of these differences we will proceed to consider anew in § 3 the possibility of removing the Goldstein singularity closer to the actual separation which is supposed to occur at $x = x_s+o(1)$.

Beforehand, however, the higher-order solution ψ_1, u_1 in the wall layer provides certain vital clues to the new length scales and structures arising in § 3, in contrast with the dominant solution ψ_0, u_0 which on its own does not immediately indicate the appropriate length scales to study. At first sight the governing equations (2.8a) for ψ_1, u_1 favour the ordering $\psi_1 = O(x_s - x)^{\frac{1}{2}}$ as $x \to x_s-$ in the lower Goldstein zone where η is $O(1)$, since the given pressure gradient in (2.8a) is of order $(x_s - x)^{-\frac{1}{2}}$ from (2.11) and ψ_0 is of order $(x_s - x)^{\frac{3}{4}}$ from (2.10). The resulting ordinary differential equation for the leading term in ψ_1 then turns out to have no solution [cf. (2.12a)–(2.14c) below] satisfying the necessary matching conditions, however, and after some trials we find that three orders of eigensolutions more singular than the forced term above have to be present to allow a self-consistent account of (2.8a–d) as $x \to x_s-$. The proposed behaviour as $x \to x_s-$ is given by

$$\psi_1 = \ln(x_s - x)\, g_{0L}(\eta) + g_0(\eta) + (x_s - x)^{\frac{1}{4}} \ln(x_s - x)\, g_{1L}(\eta) + (x_s - x)^{\frac{1}{4}} g_1(\eta) + \ldots \quad (2.12a)$$

in the lower Goldstein zone. Substitution into (2.8a) and use of (2.10), (2.11) for ψ_0, u_0, β_0 leads to the simple solution, from the $O[(x_s - x)^{-\frac{3}{4}} \ln(x_s - x)]$ terms of (2.8a),

$$g_{0L}(\eta) = -\tfrac{1}{2} A_{0L} \eta^2 \quad (2.12b)$$

for the leading term of (2.12a), where the no-slip condition (2.8b) and the requirement of no exponential growth as $\eta \to \infty$ have been applied. The constant coefficient A_{0L} remains unknown as yet. Similarly we find from (2.8a–d) with (2.10), (2.11), (2.12a, b) the next-order eigensolutions

$$g_0(\eta) = \tfrac{1}{2} A_0 \eta^2, \quad g_{1L}(\eta) = \tfrac{1}{2} A_{1L} \eta^2, \quad (2.12c)$$

where A_0, A_{1L} are unknown constants. Then, at order $(x_s - x)^{-\frac{1}{2}}$ in (2.8a), the ordinary differential equation

$$g_1''' - \tfrac{1}{8}\mu\eta^3 g_1'' + \tfrac{1}{4}\mu\eta^2 g_1' - \tfrac{1}{4}\mu\eta g_1 = -k + \alpha_0 A_{0L} \eta^2 \quad (2.13a)$$

is obtained for $g_1(\eta)$, where from the given form of β_0 in (2.11)

$$k = \mu^{-1} \alpha_0 F_s. \quad (2.13b)$$

The boundary conditions appropriate are

$$g_1(0) = g_1'(0) = 0, \quad (2.13c)$$

$$g_1(\eta) \leqslant O(\eta^2) \quad \text{as} \quad \eta \to \infty, \quad (2.13d)$$

from (2.8b) and to avoid exponential growth at infinity, respectively. The latter condition serves to fix the coefficient A_{0L} uniquely, since (2.13a) gives

$$(\eta^{-1} g_1)'' = -\eta^{-3} \exp\left(\frac{\mu\eta^4}{32}\right) \int_0^\eta (k - \alpha_0 A_{0L} \hat{\eta}^2) \exp\left(-\frac{\mu\hat{\eta}^4}{32}\right) \hat{\eta}^2 \, d\hat{\eta} \quad (2.14a)$$

when (2.13c) is applied and so we have the requirement

$$\int_0^\infty (k - \alpha_0 A_{0L} \hat{\eta}^2) \exp\left(-\frac{\mu\hat{\eta}^4}{32}\right) \hat{\eta}^2 \, d\hat{\eta} = 0 \quad (2.14b)$$

from (2.13d). Hence

$$A_{0L} = \frac{(-\frac{1}{4})!\, F_s}{2^{\frac{5}{2}} (\frac{1}{4})!\, \mu^{\frac{1}{2}}}. \quad (2.14c)$$

The presence of the leading eigensolution of (2.12a, b) is established therefore. Working to higher order would also establish the values of the unknown coefficients A_0, A_{1L} in (2.12c) and also perhaps the arbitrary multiple of η^2 which can be added even to $g_1(\eta)$.

The expansion of the solution in the upper Goldstein zone where Y is of $O(1)$ follows readily from (2.12a–c), (2.14c) and is found to be

$$\psi_1 = (x_s-x)^{-\frac{1}{2}}\mu^{-1}\psi'_{0s}(Y)\{-A_{0L}\ln(x_s-x)+A_0+A_{1L}(x_s-x)^{\frac{1}{4}}\ln(x_s-x)+A_1(x_s-x)^{\frac{1}{4}}\} + \ldots \quad (2.15)$$

on substitution into (2.8a, c) with (2.9a–c). Given the two singular behaviours of (2.12a), (2.15) as $x \to x_s-$, a comparison with (2.10), (2.9), respectively, suggests immediately that the length scale $x = x_s + O(h^{-\frac{3}{8}}\ln h)$ should be examined next, since then in (2.5a) the dominant term of $h^{-1}\psi_1$ becomes comparable with the dominant term of $h^{\frac{1}{2}}\psi_0$ and so new balances different from the classical ones of (2.2)–(2.8d) must be struck.

3. Removal of Goldstein's singularity and the appearance of a more severe one

It will be shown first, in § 3.1, that the length scale just suggested, of order $h^{-\frac{3}{8}}\ln h$ in (x_s-x), merely causes a postponement of the Goldstein singularity by means of a small shift of origin in x. This origin shift then singles out a slightly different shorter length scale, of order $h^{-\frac{3}{8}}$ in (x_s-x), for study. Second, therefore, in § 3.2 we will discuss the flow structure emerging on the shorter length scale. It is on the latter scale that the Goldstein singularity proves to be removable and in a physically sensible way, with regular separation taking place, but at the expense of a second, worse, singularity occurring further downstream within the $O(h^{-\frac{3}{8}})$ regime. The worse singularity is also physically sensible, however, and is considered further in § 4. It too proves to be removable and the basis is thereby provided for a self-consistent account of the rest of the flow development (§ 5) including the ultimate reattachment much further downstream.

3.1. *The length scale of order $h^{-\frac{3}{8}}\ln h$*

We propose that when

$$x = x_s + h^{-\frac{3}{8}}\ln(h)\,X \quad (3.1)$$

with X finite the new flow structure for (2.1a–e) has a basically two-regioned form, depicted in figure 2 and consisting of the main upper part of the oncoming wall layer, region (i), where Y is finite $[y = O(h^{-\frac{1}{2}})]$ and a lower part, region (ii), where y is only $O(h^{-\frac{5}{8}}(\ln h)^{\frac{1}{4}})$ as implied by (2.9)–(2.15). Outside the main part (i) the outer boundary condition of (2.1e) can be achieved directly. In the upper region (i) where the flow features are almost inviscid the behaviours (2.9) and (2.15) with (2.5a) lead to the expansion

$$\begin{aligned}\psi = {} & h^{\frac{1}{2}}\psi_{0s}(Y) + h^{-\frac{1}{4}}(\ln h)^{\frac{1}{2}}J_1(X,Y) + h^{-\frac{1}{4}}(\ln h)^{-\frac{1}{2}}\ln(\ln h)\,J_{1L}(X,Y) + h^{-\frac{1}{4}}(\ln h)^{-\frac{1}{2}}J_2(X,Y)\\ & + h^{-\frac{5}{8}}(\ln h)^{\frac{3}{4}}J_3(X,Y) + h^{-\frac{5}{8}}(\ln h)^{-\frac{1}{4}}\ln(\ln h)\,J_{3L}(X,Y)\\ & + h^{-\frac{5}{8}}(\ln h)^{-\frac{1}{4}}J_4(X,Y) + O(h^{-1}(\ln h)^2)\end{aligned} \quad (3.2)$$

for ψ, while the expansion indicated for p is

$$\begin{aligned}p = {} & h^2(-\tfrac{1}{2}F_s^2) + h^{\frac{5}{8}}(\ln h)(-F_sF'_sX) + h^{\frac{1}{2}}(-F_s\beta_s) + h^{-\frac{1}{4}}(\ln h)^{\frac{1}{2}}\hat{p}_1(X)\\ & + h^{-\frac{1}{4}}(\ln h)^{-\frac{1}{2}}\ln(\ln h)\,\hat{p}_{1L}(X) + h^{-\frac{1}{4}}(\ln h)^{-\frac{1}{2}}\hat{p}_2(X) + \ldots\end{aligned} \quad (3.3)$$

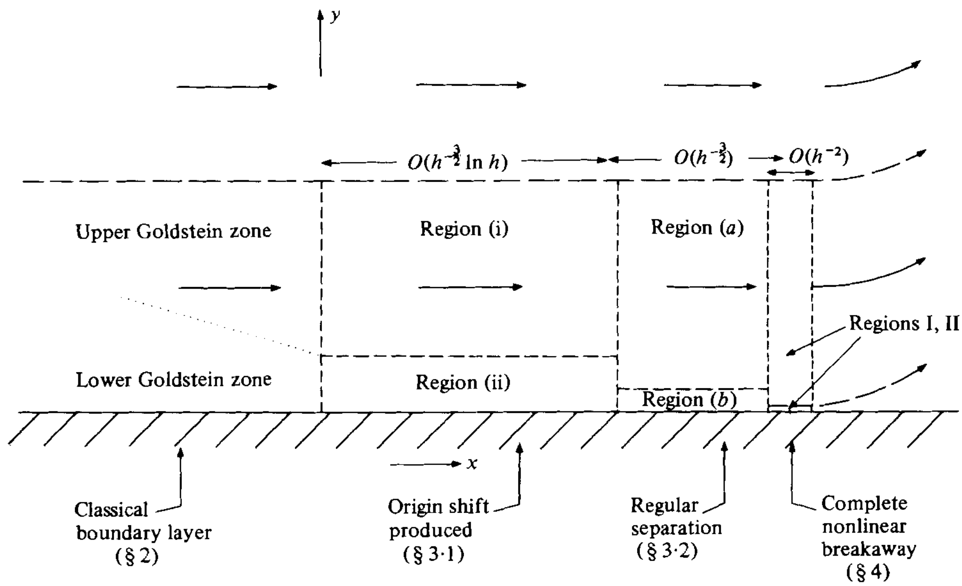

FIGURE 2. The double-structured sequence developed near $x = x_s$ by the wall layer flow as it proceeds up to and through separation (in region (*b*)) and then breaks away from the wall (via regions I, II studied in § 4). Diagram not to scale.

from (2.2), (2.4), (2.7) and (2.11). The governing equations (2.1*a*) then yield the displacement solutions

$$J_1(X, Y) = \hat{\alpha}_1(X)\,\psi'_{0s}(Y), \quad J_{1L}(X, Y) = \hat{\alpha}_{1L}(X)\,\psi'_{0s}(Y), \quad J_2(X, Y) = \hat{\alpha}_2(X)\,\psi'_{0s}(Y) \tag{3.4a}$$

for the leading terms of (3.2), where merging upstream with the classical boundary-layer form of § 2 rules out the possibility of adding arbitrary functions of Y to J_1, J_{1L}, J_2. Further terms can also be worked out, while the match with the outer constraint (2.1*e*) gives the pressure–displacement relations

$$F_s\hat{\alpha}_1(X) = \hat{p}_1(X), \quad F_s\hat{\alpha}_{1L}(X) = \hat{p}_{1L}(X), \quad F_s\hat{\alpha}_2(X) = \hat{p}_2(X) \tag{3.4b}$$

between the unknown functions of X in (3.3), (2.4*a*).

In the lower region (ii) viscous forces come into play and $y = h^{-\frac{7}{8}}(\ln h)^{\frac{1}{4}} z$ with z of $O(1)$. Again the results of (2.9)–(2.15) point to the form of the flow solution here, which is

$$\begin{aligned}\psi = {} & h^{-\frac{5}{8}}(\ln h)^{\frac{3}{4}}\left(\tfrac{1}{6}\mu z^3\right) + h^{-1}(\ln h)\,\hat{\psi}_1(X, z) + h^{-1}\ln(\ln h)\,\hat{\psi}_{1L}(X, z) \\ & + h^{-1}\hat{\psi}_2(X, z) + h^{-\frac{11}{8}}(\ln h)^{\frac{5}{4}}\,\hat{\psi}_3(X, z) + h^{-\frac{11}{8}}(\ln h)^{\frac{1}{4}}\ln(\ln h)\,\hat{\psi}_{3L}(X, z) \\ & + h^{-\frac{11}{8}}(\ln h)^{\frac{1}{4}}\,\hat{\psi}_4(X, z) + \dots\end{aligned} \tag{3.5a}$$

and is in line also with the behaviour (2.9*b*) of the dominant profile of (3.2). Substitution of (3.5*a*) into (2.1*a*), along with the pressure expansion (3.3), confirms the result (2.9*d*) at leading order. The balances at the next orders give the solutions

$$\hat{\psi}_1 = \tfrac{1}{2}\mu z^2\hat{\alpha}_1(X), \quad \hat{\psi}_{1L} = \tfrac{1}{2}\mu z^2\hat{\alpha}_{1L}(X), \quad \hat{\psi}_2 = \tfrac{1}{2}\mu z^2\hat{\alpha}_2(X), \tag{3.5b}$$

after matching with J_1, J_{1L}, J_2 as $z \to \infty$; and the governing equation

$$\tfrac{1}{2}\mu z^2 \frac{\partial^2 \hat{\psi}_3}{\partial z \partial X} - \mu z \frac{\partial \hat{\psi}_3}{\partial X} + \tfrac{1}{2}\mu^2 z^2 \hat{\alpha}_1(X)\, \hat{\alpha}_1'(X) = \frac{\partial^3 \hat{\psi}_3}{\partial z^3} \tag{3.5c}$$

for $\hat{\psi}_3$. The only allowable solution satisfying the no-slip condition at $z = 0$ has the properties

$$\hat{\psi}_3 = \tfrac{1}{2}\mu z^2 \hat{\alpha}_3(X) + \tfrac{1}{120} K z^5, \quad \mu^2 \hat{\alpha}_1(X)\, \hat{\alpha}_1'(X) = K, \tag{3.5d}$$

where the constant K can be determined from the join with the solution in the upper region (i). This join and (3.5*d*) give

$$K = -2\alpha_0^2, \quad \tfrac{1}{2}\hat{\alpha}_1^2(X) = \mu^{-2} K X + K_1 \tag{3.5e}$$

respectively, with K_1 a constant to be determined below.

Finally the properties far upstream as $X \to \infty$ must be in keeping with the Goldstein singular forms of § 2 as $x \to x_s-$ there, so that we have the asymptotes for $X \to \infty$

$$\tfrac{1}{2}\mu \hat{\alpha}_1(X) \sim \alpha_0 |X|^{\frac{1}{2}} + \tfrac{3}{4} A_{0L} |X|^{-\frac{1}{2}} + \dots, \tag{3.6a}$$

$$\tfrac{1}{2}\mu \hat{\alpha}_{1L}(X) \sim -\tfrac{1}{2} A_{0L} |X|^{-\frac{1}{2}} + \dots, \tag{3.6b}$$

$$\tfrac{1}{2}\mu \hat{\alpha}_2(X) \sim -\tfrac{1}{2} A_{0L} |X|^{-\frac{1}{2}} \ln |X| + \tfrac{1}{2} A_0 |X|^{-\frac{1}{2}} + \dots, \tag{3.6c}$$

$$\tfrac{1}{2}\mu \hat{\alpha}_3(X) \sim \alpha_1 |X|^{\frac{1}{2}} - \tfrac{3}{4} A_{1L} |X|^{-\frac{1}{2}} + \dots, \tag{3.6d}$$

after some manipulation of (2.10), (2.12*a*–*c*) with (2.5*a*) as (3.1) comes into operation. In particular, (3.6*a*) is consistent with the solution (3.5*e*) for $\tfrac{1}{2}\hat{\alpha}_1^2(X)$ provided

$$K_1 = 3\mu^{-2} \alpha_0 A_{0L} \tag{3.7a}$$

and that leaves the solution

$$\hat{\alpha}_1(X) = 2\mu^{-1} \alpha_0 \left(-X + \frac{3A_{0L}}{2\alpha_0} \right)^{\frac{1}{2}} \tag{3.7b}$$

for $\hat{\alpha}_1(X)$. Since the scaled skin friction $\hat{\tau}(X) \equiv (\partial u / \partial z)(X, 0)$ is given by

$$\hat{\tau}(X) = h^{-\frac{1}{4}} (\ln h)^{\frac{1}{4}} \mu \hat{\alpha}_1(X) \tag{3.7c}$$

to leading order, from (3.5*a*, *b*), the result (3.7*b*) suggests that the effect of the present structure and balances is merely to produce an effective origin shift in the Goldstein singularity of (2.11). The conclusion of an origin shift can be reinforced by further analysis of (3.2)–(3.6*d*) and we find the subsequent terms.

$$\left\{ \begin{aligned} &\hat{\psi}_{3L} = \tfrac{1}{2}\mu z^2 \hat{\alpha}_{3L}(X), \quad \hat{\alpha}_{1L}(X) = -\mu^{-1} A_{0L} \left(-X + \frac{3A_{0L}}{2\alpha_0} \right)^{-\frac{1}{2}}, \\ &\hat{\alpha}_2(X) = -\mu^{-1} A_{0L} \left(-X + \frac{3A_{0L}}{2\alpha_0} \right)^{-\frac{1}{2}} \ln \left(-X + \frac{3A_{0L}}{2\alpha_0} \right) + \mu^{-1} A_0 \left(-X + \frac{3A_{0L}}{2\alpha_0} \right)^{-\frac{1}{2}}, \end{aligned} \right\} \tag{3.7d}$$

where $\hat{\alpha}_{3L}(X)$ is a function of X as yet undetermined. Hence the Goldstein singularity is merely postponed and recurs at

$$X = \frac{3A_{0L}}{2\alpha_0} \quad (> 0). \tag{3.8}$$

This brings us to the next stage, which occurs when $(X - 3A_{0L}/2\alpha_0)$ is small and $O(\ln h)^{-1}$, as a comparison between the irregularities in (3.7*b*) and (3.7*d*) and reference to the expansion (3.5*a*), with (3.5*b*), indicates.

3.2. *The shorter length scale and removal of Goldstein's singularity*

Following the previous paragraph, then, we consider the shorter $O(h^{-\frac{3}{2}})$ length scale defined by

$$x = x_s + \frac{3A_{0L}}{2\alpha_0} h^{-\frac{3}{2}} \ln h + h^{-\frac{3}{2}} \bar{X} \tag{3.9}$$

with $\bar{X}$ now finite. The origin shift involved in (3.9) might have been anticipated directly from the singular properties fixed in § 2 of course, but it would seem a rather dangerous step in the theory to proceed directly from those singular properties to the length scale controlled by (3.9). Again, however, the flow needs to be studied only on two scales laterally, regions (a) and (b) of figure 2. Region(a), the inviscid upper region where Y is of order unity, has features quite similar to those of the region (i) discussed in § 3.1 and the solution develops in the displacement form (cf. (3.2), (3.4a))

$$\psi = h^{\frac{1}{2}} \psi_{0s}(Y) + h^{-\frac{1}{4}} \bar{\alpha}_1(\bar{X}) \psi'_{0s}(Y) + O(h^{-\frac{5}{8}}), \tag{3.10}$$

where $-\bar{\alpha}_1(\bar{X})$ is a function of $\bar{X}$ representing the unknown local displacement. Along with (3.10) the pressure now has the expanded form

$$p = h^2(-\tfrac{1}{2}F_s^2) - h^{\frac{1}{2}}(\ln h)(3F_s F'_s A_{0L}/2\alpha_0) + h^{\frac{1}{2}}(-F_s F'_s \bar{X} - F_s \beta_s) + h^{-\frac{1}{4}} \bar{p}_1(\bar{X}) + \dots, \tag{3.11}$$

in view of (3.3), (3.4b), (3.7b, d) and (3.9). Therefore the outer constraint (2.1e) applied as $Y \to \infty$ gives the pressure–displacement relation

$$\bar{p}_1(\bar{X}) = F_s \bar{\alpha}_1(\bar{X}) \tag{3.12}$$

(cf. (3.4b)) between the undetermined functions $\bar{p}_1$, $\bar{\alpha}_1$. This novel relation is reminiscent of the pressure-displacement laws which control separation and other interactive flow processes in external and internal motions (Stewartson 1974a, b; Messiter 1979; Smith 1977b, 1979a, b). It would be the law appropriate to hypersonic flow, indeed, but for a change of sign. The sign change is crucial, however, for two main reasons. The first is that the law (3.12) tends to prevent free interactions from occurring upstream, unlike the other laws just referred to, and this feature adds weight to the earlier comments in § 2 on the uniqueness of the solution structure ahead of separation. The second main reason will be described shortly.

Between region (a) and the wall the viscous region (b) is induced with $y = h^{-\frac{7}{8}}\bar{z}$ and $\bar{z}$ of $O(1)$. Here the solution is described by the expansion

$$\psi = h^{-\frac{5}{8}}(\tfrac{1}{6}\mu\bar{z}^3) + h^{-1}\bar{\psi}_1(\bar{X}, \bar{z}) + h^{-\frac{11}{8}}\bar{\psi}_2(\bar{X}, \bar{z}) + \dots \tag{3.13a}$$

stemming from the properties of (3.5a, b, d), (3.7b, d) as (3.9) comes into play, as well as from (3.10) with (2.9b). Substitution of (3.13a) and (3.11) into the governing equations (2.1a) again verifies the result (2.9d) at order h^2, while the terms of order $h^{\frac{13}{8}}$ produce the solution

$$\bar{\psi}_1(\bar{X}, \bar{Z}) = \tfrac{1}{2}\mu\bar{\alpha}_1(\bar{X})\bar{z}^2 \tag{3.13b}$$

when (2.1b) and the join of (3.13a) [as $\bar{z} \to \infty$] with (3.10) [as $Y \to 0$ and (2.9b) holds] are imposed. At the next order, $h^{\frac{5}{4}}$, (2.1a) requires that $\bar{\psi}_2$ satisfy the linear equation

$$\frac{\partial^3 \bar{\psi}_2}{\partial \bar{z}^3} - \tfrac{1}{2}\mu\bar{z}^2 \frac{\partial^2 \bar{\psi}_2}{\partial \bar{X}\, \partial \bar{z}} + \mu\bar{z}\frac{\partial \bar{\psi}_2}{\partial \bar{X}} = \bar{p}'_1(\bar{X}) + \tfrac{1}{2}\bar{z}^2\mu^2\bar{\alpha}_1(\bar{X})\,\bar{\alpha}'_1(\bar{X}), \tag{3.14a}$$

where (3.13b) has been used, and the boundary conditions on $\bar{\psi}_2$ are

$$\bar{\psi}_2 = \frac{\partial \bar{\psi}_2}{\partial \bar{z}} = 0 \quad \text{at} \quad \bar{z} = 0, \quad \bar{\psi}_2 \text{ not exponentially large as } \quad \bar{z} \to \infty \qquad (3.14b)$$

in the $\bar{z}$ direction, together with matching conditions in the $\bar{X}$ direction which stem from those on $\bar{\alpha}_1$ in (3.15b) below and from (3.12). Higher-order terms in the current expansions can also be considered and they seem to fall into a consistent pattern provided that (3.14a, b) are satisfied. The solution of the problem (3.14a, b) for $\bar{\psi}_2$ serves to determine $\bar{p}_1(\bar{X})$, $\bar{\alpha}_1(\bar{X})$ in fact, for the solution exists only if the right-hand side of (3.14a) satisfies a certain compatibility condition. This condition can be derived by subtracting $\mu\bar{z}\bar{\alpha}_1(\bar{X})\bar{\alpha}_1'(\bar{X})$ from $\bar{\psi}_2(\bar{X}, \bar{z})$ to give a function $f_2(\bar{X}, \bar{z})$, say, then differentiating (3.14a) with respect to $\bar{X}$ and solving the resultant linear equation and boundary conditions for $\partial \bar{f}_2/\partial \bar{X}(\bar{X}, \bar{z})$. The major steps here are described in Stewartson's (1970a, b) elegant appendix and we may appeal directly to his results, thus obtaining the condition

$$C_1 - \mu\frac{d}{d\bar{X}}(\tfrac{1}{2}\bar{\alpha}_1^2(\bar{X})) = \frac{(-\frac{1}{4})!}{2^{\frac{3}{2}}\mu^{\frac{1}{2}}(\frac{1}{4})!}\int_{-\infty}^{\bar{X}} \frac{\bar{p}_1''(\xi)\,d\xi}{(\bar{X}-\xi)^{\frac{1}{4}}} \qquad (3.15a)$$

for the existence of solutions to (3.14a, b). Here the constant C_1 is fixed by the upstream constraint

$$\bar{\alpha}_1(\bar{X}) \sim 2\mu^{-1}\alpha_0|\bar{X}|^{\frac{1}{2}} - \mu^{-1}A_{0L}|\bar{X}|^{-\frac{1}{2}}\ln|\bar{X}| + \mu^{-1}A_0|\bar{X}|^{-\frac{1}{2}} + \ldots \quad \text{as} \quad \bar{X} \to -\infty \qquad (3.15b)$$

required by matching with the solution of § 3.1 (see (3.6a–d) in particular), together with the earlier relation (3.12). Hence $C_1 = -2\mu^{-1}\alpha_0^2$.

Combining the viscous result (3.15a) and the inviscid one (3.12) therefore leaves the nonlinear integro-differential equation

$$\mu\frac{d}{d\bar{X}}(\tfrac{1}{2}\bar{\alpha}_1^2(\bar{X})) + 2\mu^{-1}\alpha_0^2 = \frac{-(-\frac{1}{4})!\,F_s}{\mu^{\frac{1}{2}}\,2^{\frac{3}{2}}(\frac{1}{4})!}\int_{-\infty}^{\bar{X}} \frac{\bar{\alpha}_1''(\xi)\,d\xi}{(\bar{X}-\xi)^{\frac{1}{4}}}, \qquad (3.15c)$$

with the incoming Goldstein form (3.15b), governing the unknown displacement $-\bar{\alpha}_1(\bar{X})$. It can be verified that the asymptote (3.15b) is consistent with the equation (3.15c), incidentally. Once $\bar{\alpha}_1(\bar{X})$ is known the pressure response $\bar{p}_1(\bar{X})$ then follows from (3.12) while the effective skin friction is $\mu\bar{\alpha}_1(\bar{X})$ from (3.13a, b); thus the displacement, the pressure perturbation and the skin friction are proportional to each other.

The problem (3.15b, c) is similar to the two discussed by Stewartson (1970a, b) in the contexts of supersonic and subsonic separation under a virtually uniform stream. In each context it was concluded that no physically sensible flow solution could result. Our case has some fundamental differences, however, connected principally with the first minus sign on the right-hand side of (3.15c) which in turn arises from our pressure–displacement law (3.12). For the sign appears to rule out the existence of eigensolutions in addition to the forced behaviour of (3.15b) far upstream (cf. Stewartson 1970a), suggesting a unique solution. Moreover even if the solution then develops a singularity at a finite value of $\bar{X}$ (cf. Stewartson 1970a) it does so in a physically acceptable fashion. Thus, the supposition that $\bar{\alpha}_1(\bar{X}) \sim m(\bar{X}_0 - \bar{X})^n$ as $\bar{X} \to \bar{X}_0-$, say, where $m, n, \bar{X}_0$ are finite but unknown and $n < \frac{3}{4}$, leads to a balance in (3.15c) only if $n = -\frac{1}{2}$ and $m = -4\mu^{-1}A_{0L}$ (see A_{0L} in (2.14c)), or $n = 1$, or $n = 0$. The first choice here raises

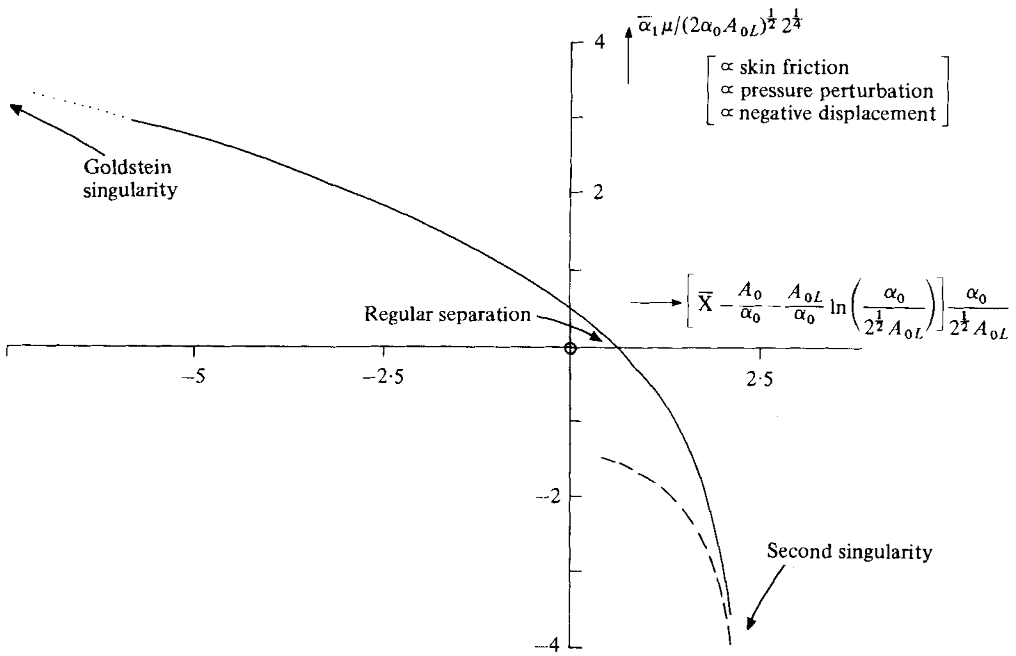

FIGURE 3. The solution (——) of (3.15*b*, *c*) for $\bar{\alpha}_1(\bar{X})$ which gives the skin friction, the pressure perturbation and the negative displacement during the removal of the Goldstein singularity in §3.2. Here the dotted line is the incoming Goldstein form (3.15*b*) and the dashed line the terminal second singularity (3.16).

the possibility that the solution of (3.15*b*, *c*) will terminate at a finite station $\bar{X} = \bar{X}_0$ with the local behaviour

$$\bar{\alpha}_1(\bar{X}) \sim -4\mu^{-1}A_{0L}(\bar{X}_0-\bar{X})^{-\frac{1}{2}} \quad \text{as} \quad \bar{X}\to\bar{X}_0- . \tag{3.16}$$

Since $A_{0L} > 0$ (3.16) implies an increasingly fast reversed flow being encountered near the wall as $\bar{X}\to\bar{X}_0-$. Hence the suggestion of (3.16) is in line with the idea that the flow solution can pass regularly through separation [at $\bar{X} = \bar{X}_s$, say, where $\bar{\alpha}_1(\bar{X}_s) = 0$ and the choice $n = 1$ just above holds], thereby removing the incoming Goldstein singularity of (3.15*b*), before moving into the singular form (3.16); for by continuity $\bar{X}_0 > \bar{X}_s$. This scenario also reinforces a belief in regularity at separation. By contrast, the pressure–displacement law of supersonic flow under a uniform stream yields an increasingly fast forward flow at the singularity of the integro-differential equation (Stewartson 1970*a*), an unrealistic phenomenon which may be expected to occur also in the corresponding hypersonic situation, where the coefficient in (3.12) and hence in (3.16) changes sign, as well as in wall jets (Smith & Duck 1977) for example.

A numerical solution of (3.15*b*,*c*) was obtained from a step-by-step centred difference treatment of the integral of (3.15*c*) with respect to $\bar{X}$, with (3.15*b*) fixing the constant of integration, and is presented in figure 3. Tests were performed on the influences of the step sizes and the upstream starting position at which (3.15*b*) was set and as a result we believe the calculation to be graphically accurate at least. The Goldstein asymptote (3.15*b*) is reproduced satisfactorily upstream and as $\bar{X}$ increases the trends towards separation indicated by (3.15*b*) continues unabated. Separation is found to take place at $\bar{X} = \bar{X}_s$, where

$$\bar{X}_s = \alpha_0^{-1}\left\{A_0 + A_{0L}\ln\left(\frac{\alpha_0}{2^{\frac{1}{2}}A_{0L}}\right) + (0{\cdot}59)\,2^{\frac{1}{2}}A_{0L}\right\}, \tag{3.17a}$$

and in accordance with the only possible choice $n = 1$ mentioned previously the solution proceeds regularly through separation. Further downstream, however, although $\bar{\alpha}_1(\bar{X})$ appears to remain unique it becomes increasingly negative as $\bar{X}$ increases and there seems to be little doubt that the termination of the flow solution proposed in (3.16) does describe accurately the numerical behaviour, as the comparison in figure 3 shows. The termination implied occurs at $\bar{X} = \bar{X}_0$, where

$$\bar{X}_0 = \bar{X}_s + (1{\cdot}76)\frac{2^{\frac{1}{2}}A_{0L}}{\bar{p}_0} \tag{3.17b}$$

according to our calculations.

We conclude therefore that the double structures of §3.1, §3.2 do act to remove the Goldstein singularity and leave the flow solution regular at separation. Beyond separation a second and worse singularity, namely (3.16), is encountered but nevertheless the physically not unrealistic behaviour of the motion associated with the second singularity encourages the view that it too can be removed, on a still shorter streamwise length scale. Accordingly we will now pursue the matter further.

4. The removal of the second singularity

The next stage of the motion takes place on the shorter length scale of order h^{-2} in x, beyond separation. The scale and the entire structure and balances promoted within it are all implied by the flow features established in the previous section and especially by the increasing speed of the reversed flow close to the wall, and the abrupt increases in displacement and in the pressure perturbation, all embodied in the singularity of (3.16).

Therefore we set

$$x = x_s + \frac{3A_{0L}}{2\alpha_0}h^{-\frac{3}{2}}\ln h + h^{-\frac{3}{2}}\bar{X}_0 + h^{-2}\hat{X}. \tag{4.1}$$

Then as before the flow structure for $\hat{X}$ of $O(1)$ consists of two regions, an upper inviscidly displaced region I comprising the majority of the oncoming wall layer and a lower viscous, but now nonlinear, region II nearer the wall. Figure 4 gives a sketch of the local flow features here.

In the upper region I the $O(1)$ lateral co-ordinate is Y again and the solution of (2.1*a*) expands in the form

$$\psi = h^{\frac{1}{2}}\psi_{0s}(Y) + \Gamma(\hat{X})\,\psi'_{0s}(Y) + \dots \tag{4.2}$$

in view of (3.10), (3.16), (4.1). Here the displacement $\Gamma(\hat{X})$ is an undetermined function of $\hat{X}$. In addition the expression for the pressure is now

$$p = h^2(-\tfrac{1}{2}F_s^2) - h^{\frac{1}{2}}(\ln h)\left(\frac{3F_sF'_sA_{0L}}{2\alpha_0}\right) + h^{\frac{1}{2}}(-F_sF'_s\bar{X}_0 - F_s\beta_s) + \hat{p}_1(\hat{X}) + \dots \tag{4.3}$$

as implied by (3.11), (3.12) and (3.16) with (4.1). So the outer boundary condition (2.1*e*)

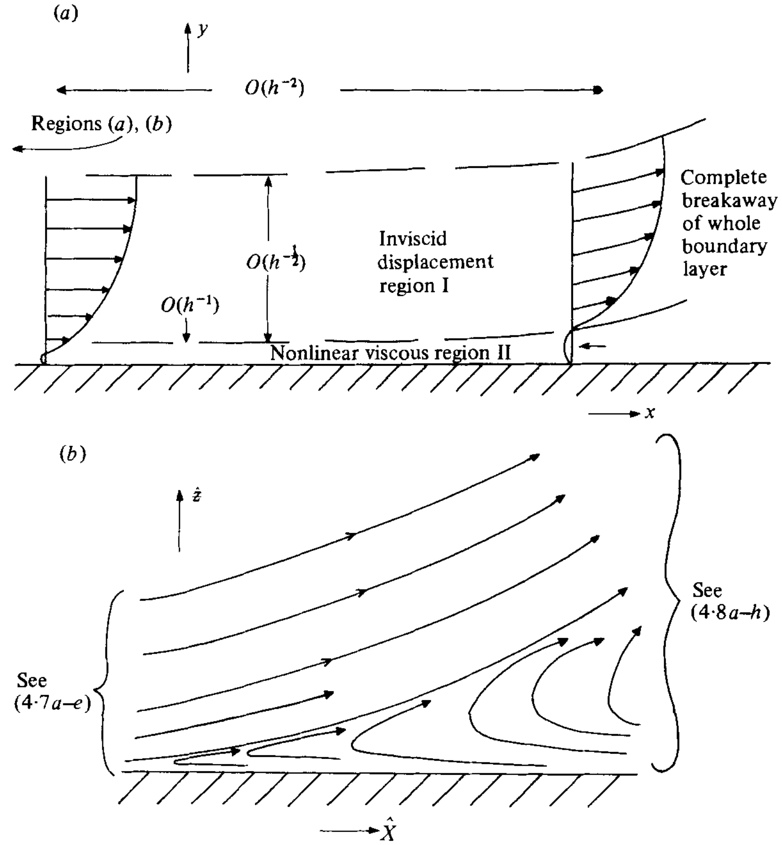

FIGURE 4. (a) Schematic diagram (not to scale) of the double structure I–II of §4 required to smooth out the singularity (3.16) and to complete the breakaway of the wall-layer flow. (b) Sketch of the streamline pattern induced in region II near the wall.

is satisfied by (4.2), (4.3) provided the new pressure–displacement law

$$\hat{p}_1(\hat{X}) = F_s\Gamma(\hat{X}) - F_s F_s'\hat{X} \tag{4.4}$$

holds, on use of (4.1). The law (4.4) connecting the unknown functions $\hat{p}_1(\hat{X})$, $\Gamma(\hat{X})$ contrasts with the earlier local laws of (3.4*b*), (3.12). Here the locally uniform adverse pressure gradient $\propto F_s F_s'$, which caused the original oncoming wall layer to approach separation in the first place, suffers a finite change in character as opposed to the infinitesimal changes that occurred further upstream (see (3.3), (3.11)). On the other hand the present law like the earlier ones can be considered to be a free perturbation of the original relation (2.4) which in turn is just a Bernoulli balance.

The flow solution in the lower region II follows from the properties of (4.2) with (2.9*b*) and has the development

$$\psi = h^{-1}\hat{\psi}(\hat{X}, \hat{z}) + \dots. \tag{4.5}$$

Here $y = h^{-1}\hat{z}$ and $\hat{z}$ is of order unity. Hence substitution into (2.1*a*) reproduces the classical boundary-layer equations

$$\hat{u} = \frac{\partial\hat{\psi}}{\partial\hat{z}}, \quad \hat{u}\frac{\partial\hat{u}}{\partial\hat{X}} - \frac{\partial\hat{\psi}}{\partial\hat{X}}\frac{\partial\hat{u}}{\partial\hat{z}} = -\hat{p}_1'(\hat{X}) + \frac{\partial^2\hat{u}}{\partial\hat{z}^2} \tag{4.6a}$$

governing $\hat{\psi}(\hat{X}, \hat{z})$. In addition the boundary conditions become

$$\begin{cases} \hat{\psi} = \hat{u} = 0 \quad \text{at} \quad \hat{z} = 0, & (4.6b) \\ \hat{\psi} \sim \frac{1}{6}\mu(\hat{z} + \Gamma(\hat{X}))^3 \quad \text{as} \quad \bar{z} \to \infty, & (4.6c) \\ \hat{\psi} \to \frac{1}{6}\mu\hat{z}^3 \quad (0 < \hat{z} < \infty) \quad \text{as} \quad \hat{X} \to -\infty, & (4.6d) \end{cases}$$

from (2.1b) and matching with (4.2) in region I and with (3.13a) in the region (b) upstream, respectively; a manipulation of (4.2), (2.9) consistent with the governing equation (4.6a) has been used to establish (4.6c) here. The flow response on the present short length scale is therefore dictated by the solution properties of the nonlinear interactive problem (4.6a–d) together with the law (4.4). This problem, like the linear ones discussed before, bears a resemblance to the other interactive problems encountered in quite different contexts (see Stewartson 1974a, b; Messiter 1979; Smith 1977a, b, 1979a–c; Smith & Duck 1977) in internal and external flows, but it also has some interesting unique aspects which can be seen below in the far upstream and downstream character envisaged for the local motion.

First, far upstream as $\hat{X} \to -\infty$ the join of the pressures in (4.3) and (3.11) requires, from (3.12), (3.16),

$$\hat{p}_1(\hat{X}) \sim -F_s F_s' \hat{X} - 4\mu^{-1} F_s A_{0L} |\hat{X}|^{-\frac{1}{2}} + \ldots \tag{4.7a}$$

so that from (4.4) the scaled displacement also diminishes algebraically,

$$\Gamma(\hat{X}) \sim -4\mu^{-1} A_{0L} |\hat{X}|^{-\frac{1}{2}} + \ldots, \tag{4.7b}$$

which is consistent with the singularity of (3.16). Then (4.6a) retrieves a linear form, since

$$\hat{\psi}(\hat{X}, \hat{Z}) \sim \tfrac{1}{6}\mu |\hat{X}|^{\frac{3}{4}} \hat{\eta}^3 + f(\hat{\eta}) + \ldots, \tag{4.7c}$$

where $\hat{\eta} = \hat{z} |\hat{X}|^{-\frac{1}{4}}$ is $O(1)$. Substitution of (4.7a–c) into (4.6a) and application of the constraints (4.6b, c) yields the solution

$$f(\hat{\eta}) = -2A_{0L} \hat{\eta}^2 \tag{4.7d}$$

for the perturbation in (4.7c). Hence the scaled skin friction $\hat{\tau}_w \equiv \partial\hat{u}(\hat{X}, 0)/\partial\hat{z}$ tends to zero upstream but is negative, in the form

$$\hat{\tau}_w(\hat{X}) \sim -4A_{0L} |\hat{X}|^{-\frac{1}{2}} \quad \text{as} \quad \hat{X} \to -\infty \tag{4.7e}$$

which, again, is consistent with the scaled skin friction $\mu\bar{\alpha}_1(\bar{X})$ of § 3.2, from (3.16) and (4.1). A comparison between (4.7e) and (4.6d) shows that the far upstream starting profile for the solution of (4.6a–d), (4.4) is already a separated one, an unusual feature but in keeping with the conclusions of § 3.2. The dividing streamline $\hat{\psi} = 0$ starts off along the curve $\hat{z} \sim 12\mu^{-1} A_{0L} |\hat{X}|^{-\frac{1}{2}}$ from (4.7c, d).

Second, assuming that the flow solution of (4.6a–d) remains regular as $\hat{X}$ increases from $-\infty$, we see that a self-consistent account of its far downstream behaviour can also be advanced. The slight diminishing far upstream of the incoming strong adverse pressure gradient in (4.7a) and the slight rise of the displacement in (4.7b) are believed to herald the start of a nonlinear process in which the pressure gradient continues to fall and the displacement to rise, because of (4.4). Let us suppose that as a result the

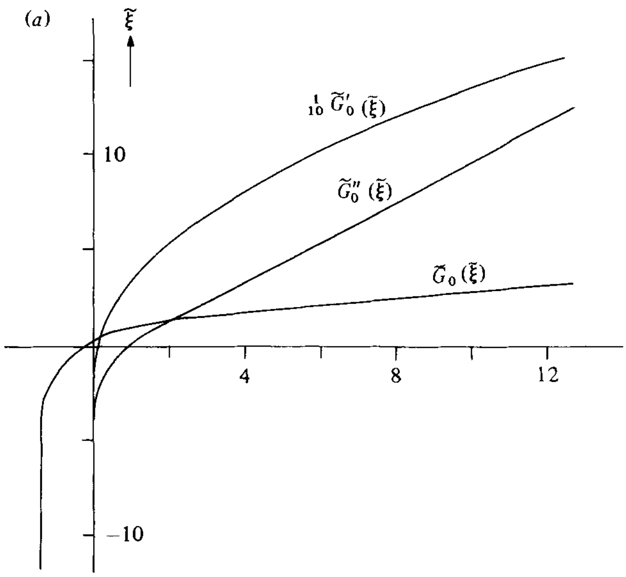

FIGURE 5(*a*). For the caption see next page.

pressure gradient ultimately tends to zero, from above, as $\hat{X} \to +\infty$, with the pressure levelling out at an unknown constant value $\hat{p}_{1\infty}$. Then (4.4) requires

$$-\Gamma(\hat{X}) \sim -F_s'\hat{X} - F_s^{-1}\hat{p}_{1\infty}, \tag{4.8a}$$

$$\hat{p}_1 \to \hat{p}_{1\infty} \quad \text{as} \quad \hat{X} \to \infty, \tag{4.8b}$$

so that the displacement $-\Gamma(\hat{X})$ increases linearly with $\hat{X}$. This effective breakaway of the wall layer motion and the simultaneous levelling out of the pressure are both in line with the flow of (4.6*a–d*) tending to concentrate in a detached expanding shear layer surrounding the dividing streamline $\hat{z} \sim -F_s'\hat{X}$ far downstream, leaving only a slower reversed motion underneath, between the shear layer and the wall, while above the shear layer the rest of the wall layer responds passively, retaining its original profile but displaced by a relative amount $-h^{-\frac{1}{2}}\Gamma(\hat{X})$ according to (4.2). Thus the proposition for $\Gamma(\hat{X})$ in (4.8*a*) reflects a breakaway of the majority of the original wall layer. The associated asymptotic solution is similar to those of Stewartson & Williams (1973), Smith (1977*a*, *b*, 1978), Smith & Duck (1977). It is controlled by the $O(\hat{X}^{\frac{1}{3}})$ thick shear layer, wherein $\hat{z} = -\Gamma(\hat{X}) + \hat{X}^{\frac{1}{3}}\xi$ with ξ of $O(1)$ and

$$\hat{\psi} \sim \hat{X}^{\frac{2}{3}}G_0(\xi) + \dots \quad \text{as} \quad \hat{X} \to \infty. \tag{4.8c}$$

From (4.6*a–c*) and (4.8*a–c*) G_0 satisfies the nonlinear similarity equation and constraints

$$\left.\begin{aligned} &G_0''' + \tfrac{2}{3}G_0G_0'' - \tfrac{1}{3}G_0'^2 = 0,\\ &G_0 \sim \tfrac{1}{6}\mu\xi^3 + 0 \quad \text{as} \quad \xi \to \infty,\\ &G_0'(-\infty) = 0, \end{aligned}\right\} \tag{4.8d}$$

where the second constraint anticipates the slowness of the reversed flow beneath the detached shear layer. The solution of (4.8*d*) found numerically is shown in figure 5 and has the property

$$G_0(-\infty) = -C_0 = -1{\cdot}39\mu^{\frac{1}{3}}. \tag{4.8e}$$

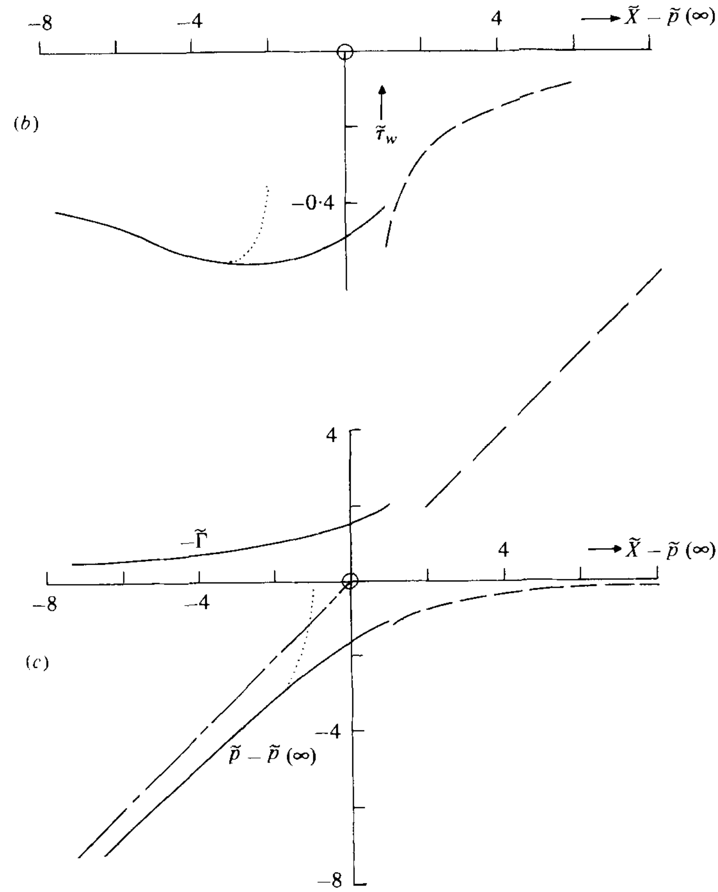

FIGURE 5. (a) The similarity solution (4.8d). Here $\tilde{G}_0 \equiv \mu^{-\frac{1}{4}}G_0$, $\tilde{\xi} \equiv \mu^{\frac{1}{4}}\xi$. (b) The numerical solution (——) of (4.10a–e) for τ_w, ρ, Γ versus $\tilde{X}$, together with the upstream asymptotic series solution (......) given in (4.11a)–(4.12c), the downstream asymptotes (4.8a, g, h) (– – –) and the upstream asymptote (——–—) of (4.7a). The constant $p(\infty)$ remains arbitrary in the solution.

Hence beneath the shear layer only a weak uniform backward stream is provoked, with

$$\hat{\psi} \sim -(-F_s')^{-1}C_0\hat{z}\hat{X}^{-\frac{1}{4}}, \quad \hat{u} \sim -(-F_s')^{-1}C_0\hat{X}^{-\frac{1}{4}}, \tag{4.8f}$$

from (4.8e). Here (4.8f) is consistent with (4.6a) provided

$$\hat{p}_1 \sim \hat{p}_{1\infty} - \frac{1}{2}\left(\frac{C_0}{F_s'}\right)^2 \hat{X}^{-\frac{1}{2}} \tag{4.8g}$$

(see (4.8b)). Also, a thinner viscous reversed sublayer of thickness $O(\hat{X}^{\frac{3}{8}})$ is provoked between the stream of (4.8f) and the wall and its similarity solution obtained numerically predicts the behaviour

$$\hat{\tau}_w(\hat{X}) \sim -0{\cdot}528(-F_s'^3F_s^{-1})^{-\frac{3}{8}}\hat{X}^{-\frac{7}{8}} \tag{4.8h}$$

for the scaled skin friction far downstream.

Both the proposed forms (4.7a–e) and (4.8a–h) far upstream and far downstream give self-consistent and realistic behaviours for the motion, therefore. Judging from those forms one might expect the motion to remain separated throughout the present stage of its development and to become increasingly so, as the downstream distance increases, in the sense of the dividing streamline being forced further from the wall but with the strength of the reversed flow velocities eventually decreasing, according to (4.8*f*, *h*), after an initial incremental stage implied by (4.7*c*–*e*).

Due to the nonlinearity of (4.6*a*), however, a numerical solution of the fundamental problem (4.6*a*–*d*) with (4.4) is necessary, to verify the above expectations among other things. Bearing in mind the anticipated presence of separated flow near the wall for all $\hat{X}$, we tried several computational approaches intended to allow for both forward and backward iterative sweeping of the discretized flow domain; but remarkably enough it was found that a fairly straightforward centred-difference approach involving only downstream marching instead seemed to provide a satisfactory numerical integration of the system. First we set

$$\hat{\psi} = (-F_s')^{-1}\tilde{\psi}, \quad \hat{p}_1 = F_s^{\frac{2}{3}}(-F_s')^{-\frac{2}{3}}\tilde{p}, \quad \Gamma = (F_s F_s'^2)^{-\frac{1}{3}}\tilde{\Gamma},$$
$$\hat{X} = F_s^{-\frac{1}{3}}(-F_s')^{-\frac{5}{3}}\tilde{X}, \quad \hat{z} = (F_s F_s'^2)^{-\frac{1}{3}}\tilde{z} \tag{4.9}$$

to leave the fundamental problem in the form

$$\tilde{u} = \partial\tilde{\psi}/\partial\tilde{z}, \quad \tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{X}} - \frac{\partial\tilde{\psi}}{\partial\tilde{X}}\frac{\partial\tilde{u}}{\partial\tilde{z}} = -\tilde{p}'(\tilde{X}) + \frac{\partial^2\tilde{u}}{\partial\tilde{z}^2}, \tag{4.10a}$$

$$\tilde{\psi} = \tilde{u} = 0 \quad \text{at} \quad \tilde{z} = 0, \tag{4.10b}$$

$$\tilde{\psi} \sim \tfrac{1}{6}(\tilde{z} + \tilde{\Gamma}(\tilde{X}))^3 \quad \text{as} \quad \tilde{z} \to \infty, \tag{4.10c}$$

$$\tilde{\psi} \sim \tfrac{1}{6}\tilde{z}^3 \quad (0 < \tilde{z} < \infty) \quad \text{as} \quad \tilde{X} \to -\infty, \tag{4.10d}$$

$$\tilde{p}(\tilde{X}) = \tilde{\Gamma}(\tilde{X}) + \tilde{X}, \tag{4.10e}$$

free of the constants F_s, F_s', μ, using (2.9*d*). Then the infinite series solution developing from the leading terms (4.7*a*–*d*) upstream was considered, in order to gain some guidance on the numerical properties of the solution. The series solution takes the form

$$\tilde{\psi} = \tfrac{1}{6}\tilde{\eta}^3|\tilde{X}|^{\frac{2}{3}} + \sum_{n=1}^{\infty}[q_n(\tilde{\eta}) - \tfrac{1}{2}\pi_n\tilde{\eta}^2]|\tilde{X}|^{\frac{2}{3}(1-n)}, \tag{4.11a}$$

$$\tilde{p} = \tilde{X} - \sum_{n=1}^{\infty}\pi_n|\tilde{X}|^{\frac{1}{3}(1-3n)}, \tag{4.11b}$$

where $\tilde{\eta} = \tilde{z}|\tilde{X}|^{-\frac{1}{3}}$ and the functions $q_n(\tilde{\eta})$ and the constants π_n are to be found, but $q_1(\tilde{\eta}) \equiv 0$. Again, and surprisingly for a reversed flow, eigensolutions in addition to the terms of (4.11*a*, *b*) far upstream do not appear to be possible mainly because of the signs present in the pressure–displacement law (4.4), or (4.10*e*) just as in § 3.1. Substitution of (4.11*a*, *b*) into (4.10*a*–*e*) formally leaves $q_n(\tilde{\eta})$, π_{n-1} controlled by the linear ordinary differential equation and boundary conditions, for $n \geqslant 2$,

$$q_n''' - \tfrac{1}{8}\tilde{\eta}^3 q_n'' + \tfrac{1}{8}(4-3n)\tilde{\eta}^2 q_n' + \tfrac{3}{4}(n-1)\tilde{\eta}q_n = \tfrac{1}{4}(4-3n)\pi_{n-1} + Q_n(\tilde{\eta}), \tag{4.12a}$$

$$q_n(0) = q_n'(0) = 0, \quad q_n''(\infty) = 0. \tag{4.12b}$$

Here the forcing term $Q_n(\tilde{\eta})$ depends on contributions of lower order than q_n, π_{n-1}, since

$$Q_n(\tilde{\eta}) = \frac{1}{4}\sum_{j=1}^{n-1} [(3n-5)(q_j' - \tilde{\eta}\pi_j)(q_{n-j}' - \tilde{\eta}\pi_{n-j}) - 3(n-2)(q_j'' - \pi_j)(q_{n-j} - \tfrac{1}{2}\tilde{\eta}^2\pi_{n-j})] \tag{4.12c}$$

for $n \geqslant 2$, so that (4.12a–c) allow the successive determination of the pairs (q_n, π_{n-1}) for $n \geqslant 2$. It can be shown in particular that the solution of (4.12a–c) for $n = 2$ implies the result $\pi_1 = (-\frac{1}{4})!/2^{\frac{1}{2}}(\frac{1}{4})!$ consistent with upstream asymptotes (4.7a–e), from (2.14c). Calculations of (4.12a–c) were carried out for the first twenty terms of the series and are summarized in figure 5. Checks were also made on the effects of the step sizes used, the curtailment of the integration domain at a large value of $\tilde{\eta}$ and the tolerances set in the iterative scheme required to solve for $q_2(\tilde{\eta})$, and as a result we believe the calculated solutions to be satisfactory. There is a strong indication that the series solution may have a not insubstantial radius of convergence in terms of $|\tilde{X}|^{-1}$. Obviously it is divergent at $\tilde{X} = 0$, but when we worked in terms of a transformed co-ordinate χ defined by $\chi(1-\chi^{\frac{3}{2}})\tilde{X} = (2\chi^{\frac{3}{2}} - 1)$ (where $0 < \chi < 1$), in the hope that a series solution in powers of χ might prove convergent over an extended domain, the same features as in figure 5 were found to hold. One major point of the series solution above is that it gives a very good idea of the solution properties. In particular it helped in the making of an accurate first guess in the iterative marching routine used subsequently to solve (4.10a–e). Here, given a guess for the displacement function $\tilde{\Gamma}(\tilde{X})$ consistent with (4.7b), (4.8a), the boundary-layer problem of (4.10a–d) only was integrated forward, thus determining the corresponding pressure response $\tilde{p}(\tilde{X})$. Then (4.10e) was invoked to yield a new guess for $\tilde{\Gamma}(\tilde{X})$ and so the iterative marching could be repeated until convergence was achieved. Again, checks were made on the influences of step sizes, of the outer and upstream boundary conditions, and of the tolerance in the Newton iterative scheme used at each discrete step. The routine was based on Smith's (1972, 1974). The numerical results for $\tilde{\Gamma}(\tilde{X})$, $\tilde{p}(\tilde{X})$ and the effective skin friction $F_s^{-\frac{3}{4}}(-F_s')^{-\frac{1}{4}}\hat{\tau}_w \equiv \tilde{\tau}_w(\tilde{X}) \equiv \partial\tilde{u}/\partial\tilde{z}(\tilde{X}, 0)$ are displayed in figure 5. Although the forward marching scheme eventually proved divergent, or at least very slowly convergent, it did so only after a long traverse of the flowfield had been accomplished and by then a distinctive trend towards the downstream behaviours of (4.8a–g) had already emerged, as the comparisons in figure 5 show.

So the numerical evidence strongly supports the view that during the present short scale stage the flow adjusts smoothly from the incoming form (4.7a–e) far upstream to the outgoing form (4.8a–h) far downstream. The singularity of (3.16) is thereby removed.

As the motion proceeds further downstream the entire original wall layer is forced to climb away from the wall (see (4.8a)) as the pressure achieves its plateau, in (4.8b), and the reversed flow velocities diminish in strength. These features form the basis for the wake properties, the rest of the flow development, including the ultimate reattachment process which takes place over a vast length scale beyond the present stage and will be described in § 5.

5. The remainder of the flow development, and reattachment

The viscous shear layer emerging at almost constant pressure on the downstream side of the nonlinear stage just studied is fairly concentrated around the straight dividing streamline $Y = h^{-\frac{1}{2}}(-F_s')\hat{X}$, and the entire wall layer of thickness $O(h^{-\frac{1}{2}})$ passively follows the same displacement curve, in the sense that above the shear layer

$$\psi \sim h^{\frac{1}{2}}\psi_{0s}[Y - h^{-\frac{1}{2}}(-F_s')\hat{X}] \quad \text{as} \quad \hat{X} \to \infty \tag{5.1}$$

from (4.2), (4.8a). On the other hand the shear layer spreads out with increasing distance downstream. Consequently the simple detachment effect of (5.1) continues undisturbed until the thickness $O(\hat{X}^{\frac{1}{3}}h^{-1})$ of the shear layer becomes comparable with the detached wall layer thickness $O(h^{-\frac{1}{2}})$ of (5.1), i.e. until $\hat{X}$ reaches $O(h^2)$, or from (4.1), until $x - x_s = O(1)$. When the latter stage is reached the whole detached wall layer alters and at the least must start to entrain fluid from underneath. Further we propose that the flow beneath the detached viscous layer of thickness $O(h^{-\frac{1}{2}})$ then remains relatively slow, although not necessarily dominated solely by the need to supply the small entrainment into the viscous layer. This means that the pressure at and below the viscous layer stays virtually uniform throughout the stage where $x - x_s$ is $O(1)$ and positive, and so the present proposal corresponds to the extension of the free streamline approach which has been applied elsewhere to internal and external separating flows. In internal flows (Smith 1978, 1979a; Smith & Duck 1980) the extended free streamline approach is found to lead to a self-consistent account of grossly separated flows at high Reynolds numbers, even when reattachment eventually occurs. The same is now believed to be true in external flows also (Smith & Stewartson 1973; Burggraf 1975; Messiter 1975, 1979; Smith 1979b) although, unlike in the internal case, a consistent description of reattachment downstream has remained elusive so far in the external case, with one exception (Smith & Stewartson 1973; Stewartson 1974b). The present flow situation can be regarded as an example of either case since it applies equally to internal and external motions over obstacles (Smith 1976a, b; Smith *et al.* 1981). Nevertheless, the following indicates that due primarily to the action of viscosity in the eventual reattachment process the extended free streamline approach proposed here does lead to a self-consistent account of the grossly separated motion which develops during and beyond the stage $x - x_s = O(1)$ currently under consideration.

During the current stage, then, the thin detached viscous layer stays concentrated around the dividing streamline $\psi = 0$, which starts as the straight line

$$y \sim h(-F_s')(x - x_s)$$

in view of (5.1), with (4.1) and $Y = h^{\frac{1}{2}}y$, to leading order. Indeed the dividing streamline must continue along the path $y = y_d(x)$, where

$$y_d(x) = h[F_s - F(x)] \tag{5.2a}$$

throughout the current stage in order to preserve the uniformity of the pressure at and below the thin viscous layer surrounding the dividing streamline. For above the

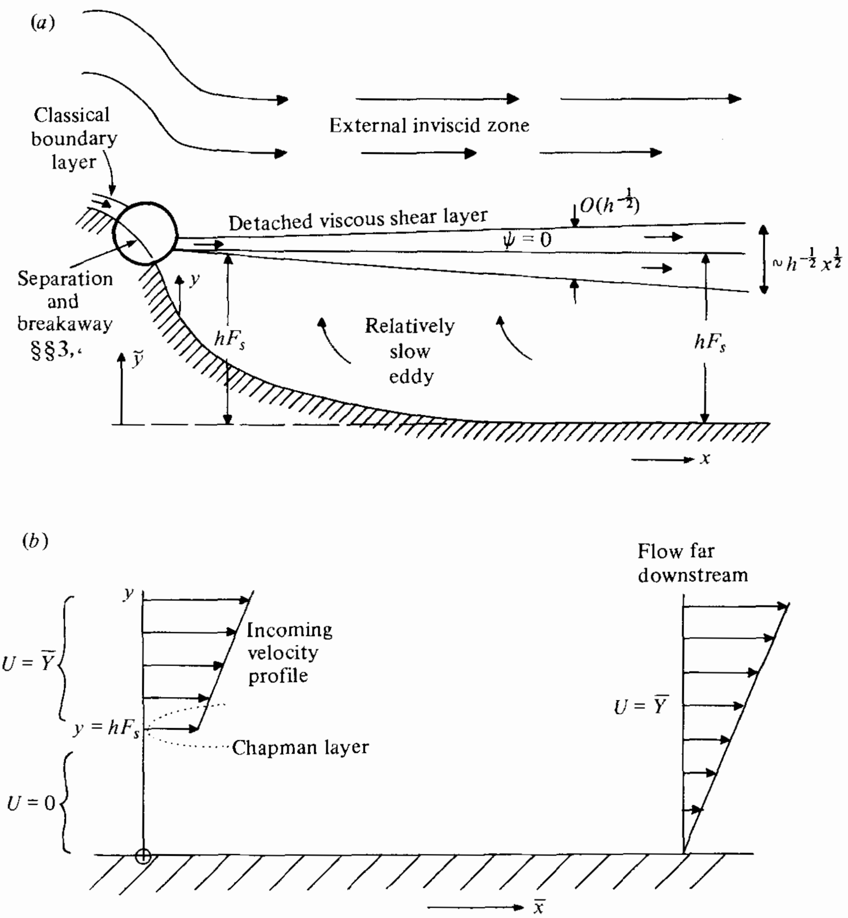

FIGURE 6. (a) Sketch of the separated flow structure (§5) mainly beyond the separation at $x = x_s$. The structure remains intact up to distances $O(h^3)$ downstream. (b) The main features of the reattachment and final phase of the motion at distances $O(h^3)$ downstream.

dividing streamline the flow solution continues to take the inviscid form of (2.2), (2.3*a*) to leading order, consistent with the outer boundary condition (2.1*d, e*) with $y = h\bar{y}$; but at the dividing streamline the requirements that $\psi = 0$ and $p_0(x)$ be uniform then yield (5.2*a*) to within a constant, which can be evaluated from continuity of pressure or streamline position at $x = x_s \pm$. Along with (5.2*a*) therefore we have the pressure level

$$p = -\tfrac{1}{2}h^2 F_s = \text{constant} \tag{5.2b}$$

for $x > x_s$, to leading order. An alternative view of (5.2*a*) in terms of the untransformed distance $\tilde{y} = y + h\,F(x)$ (see §2) gives the straight line

$$\tilde{y} = hF_s \tag{5.2c}$$

for the shape of the dividing streamline, in keeping with Smith's (1978, 1979*a*) conclusions (see also §6). Beneath the $O(h^{-\frac{1}{2}})$ viscous shear layer astride (5.2*a, c*) the fluid is only slowly moving. If it merely entrains into the shear layer, which is an eminent possibility, that would suggest that u, ψ are only $O(h^{-\frac{1}{2}})$ and $O(h^{\frac{1}{2}})$ there, but an eddying flow with velocities satisfying $h^{-\frac{1}{2}} \ll u \ll h$ is also supportable. A match with the emergent properties (4.8*f*–*h*) closer to separation as $x \to x_s +$ is readily achieved (Smith 1979*b*).

The flow within the shear layer is governed by the classical boundary-layer equations on the $x - x_s = O(1)$ scale, essentially with a uniform speed $u = hF_s$ just above and negligible speed below. It starts at $x = x_s +$ in the form (5.1), except for the viscous modification in (4.8*c*–*e*) there, and continues to entrain fluid as $x - x_s$ increases.

The balance above (see also figure 6) continues to hold over an enormous downstream range, until the flow enters its final phase at $O(h^3)$ distances downstream. For the shear-layer thickness, although $O(h^{-\frac{1}{2}})$ for x of $O(1)$, expands like $x^{\frac{1}{2}}$ in the Chapman form far downstream and so becomes an appreciable fraction of the $O(h)$ distance (in (5.2*a*)) from the wall only as the $x = O(h^3)$ stage is reached. This makes use of the assumed property $F(\infty) = 0$ for the obstacle shape, although other far-field properties could be accommodated. In the final phase u, ψ, p, x, y have the respective orders h, h^2, h^2, h^3, h implied by the behaviour of the previous stage as $x \to \infty$, in (5.2*a*, *b*) for example or in the Chapman form (see below). Hence from (2.1*a*) the dominant motion is controlled by the boundary-layer equations

$$U = \frac{\partial \Psi}{\partial \bar{Y}}, \quad U\frac{\partial U}{\partial \bar{x}} - \frac{\partial \Psi}{\partial \bar{x}}\frac{\partial U}{\partial \bar{Y}} = -\frac{dP}{d\bar{x}} + \frac{\partial^2 U}{\partial \bar{Y}^2} \tag{5.3}$$

throughout its final phase, where

$$(u, \psi, p, x, y) = (hU, h^2\Psi, h^2P, h^3\bar{x}, h\bar{Y}) + \dots. \tag{5.4}$$

Starting conditions for (5.3) are

$$\begin{cases} \Psi \to \tfrac{1}{2}(\bar{Y}^2 - F_s^2), \quad U \to \bar{Y} \quad \text{for} \quad \bar{Y} > F_s \quad \text{as} \quad \bar{x} \to 0+, & (5.5a) \\ \Psi, U \to 0 \quad \text{for} \quad \bar{Y} < F_s \quad \text{as} \quad \bar{x} \to 0+, & (5.5b) \\ P(0) = -\tfrac{1}{2}F_s^2. & (5.5c) \end{cases}$$

Here (5.5*a*) joins the present solution with the previous uniform shear form (2.3*a*, *b*) holding above the incoming shear layer at $\bar{Y} = F_s$, by use of (5.2*a*), while (5.5*b*) stems from the absence of any significant motion below the shear layer up to the present stage and (5.5*c*) matches the pressure with (5.2*b*). The boundary conditions beyond the start of (5.5*a*–*c*) are

$$\begin{cases} U = \Psi = 0 \quad \text{at} \quad \bar{Y} = 0, & (5.5d) \\ U - \bar{Y} \to 0, \quad \Psi \sim \tfrac{1}{2}\bar{Y}^2 + P(\bar{x}) \quad \text{as} \quad \bar{Y} \to \infty, & (5.5e) \end{cases}$$

from (2.1*b*, *d*, *e*), since $F(\infty) = 0$. The pressure $P(\bar{x})$ remains to be determined as in the rather different problems of reattaching flow addressed by Jenson (1975), Smith (1979*a*) and Daniels (1979).

The fundamental nonlinear problem (5.3), (5.5*a*–*e*) governing the final flow development requires a numerical solution. The main complications involved concern the strong singularity at the start of the flow and the presence and nature of the reversed flow immediately afterwards as an eddy of recirculating fluid below the dividing streamline $\Psi = 0$ is set significantly into motion at last. The singularity occurs because as $(\bar{x}, \bar{Y}) \to (0+, F_s+)$ the Chapman form

$$\Psi \sim \bar{x}^{\frac{1}{2}} f_c(\zeta), \quad \text{where} \quad \zeta = (\bar{Y} - F_s)\bar{x}^{-\frac{1}{2}}, \tag{5.6a}$$

referred to previously is recovered, from (5.5a,b). Here from (5.3), (5.5a,b), $f_c(\zeta)$ satisfies the Blasius equation $2f_c''' + f_c f_c'' = 0$ but with $f_c'(\infty) = F_s$, $f_c'(-\infty) = 0$ and the solution has the property

$$f_c(-\infty) = -F_s^{\frac{1}{2}}\kappa, \qquad (5.6b)$$

where $\kappa = 1{\cdot}24$. This property necessitates the existence of some slow reversed flow between $\bar{Y} = 0$, $\bar{Y} = F_s-$ as $\bar{x} \to 0+$, from (5.5b,d). A uniform reversed stream like (4.8f) but $\propto \bar{x}^{\frac{1}{2}}$ is inappropriate here, however, because between the stream and the wall the viscous sublayer required locally for the satisfaction of the no-slip condition has no corresponding similarity solution. This is perhaps not surprising in view of the deceleration required of the reversed flow by (5.5b). In fact it prompts the suggestion that the motion near the wall must be forward, which in turn implies the presence of at least two eddies of recirculating fluid beyond $\bar{x} = 0$, as in Smith's (1979a) study. In addition the existence of eigensolutions inside and outside the incoming shear-layer region of (5.6a,b) apparently cannot be ruled out on the grounds of local consistency alone. Thus on the face of it the slowish starting flow beneath the incoming shear layer can be expressed in a number of ways, including the form

$$\Psi = \bar{x}^n A_n \sin(\lambda \bar{Y}) + \ldots, \quad P = -\tfrac{1}{2}F_s^2 - \tfrac{1}{2}A_n^2\lambda^2\bar{x}^{2n} + \ldots \quad (0 < \bar{x} \ll 1) \qquad (5.6c)$$

with the unknown constants λ, A_n, n satisfying $\lambda > 0$, $A_n > 0$, $0 < n \leqslant \frac{1}{2}$, from (5.5$b$), (5.6$b$) and above. If $n < \frac{1}{2}$ then λ must be a multiple of πF_s^{-1} from (5.7b) and almost certainly an *even* multiple if a consistent match is to be achieved between the shear layer and the inviscid region below. If $n = \frac{1}{4}$ the normal velocity generated above the shear layer is just sufficient to provoke a finite displacement of the Chapman profile on the scale $\zeta = O(1)$, the full boundary condition for f_c at $\zeta = \infty$ then being

$$f_c \sim F_s\zeta - \tfrac{1}{2}A_{\frac{1}{4}}^2\lambda^2 \quad (\zeta \to \infty), \qquad (5.6d)$$

whereas for $\frac{1}{4} < n \leqslant \frac{1}{2}$ the finite part of f_c must vanish as $\zeta \to \infty$. For $n = \frac{1}{2}$ we also require $A_n \sin \lambda F_s = -F_s^{\frac{1}{2}}\kappa$ so that then $A_n > F_s^{\frac{1}{2}}\kappa$ and $\lambda > \pi F_s^{-1}$.

Neither physical intuition nor Proudman's (1960) suggestions seem to tie down the values of the unknown coefficients arising locally, although the entrainment-dominated eddy flow on the $x = O(1)$ length scale, corresponding to $n = \frac{1}{2}$, would appear to be the only one which provides a direct match with the flow features (4.8f,g) just beyond separation. Presumably the need for a global solution of (5.3), (5.5a–e), which we would expect to yield the behaviour

$$\Psi \to \tfrac{1}{2}Y^2, \quad U \to \bar{Y}, \quad P \to 0 \quad \text{as} \quad \bar{x} \to \infty \qquad (5.7)$$

as the original forward flow to (2.1c) is retrieved far downstream, must place some restriction on the possible eigensolutions occurring in the initial phase although whether the solution to (5.3), (5.5a–e) and (5.7) is unique remains open to question. However, there seems to be little doubt that a solution exists, as in Smith's (1978, 1979a) cases, and a preliminary numerical investigation of its properties was undertaken. This concentrated primarily on the importance of a satisfactory treatment of the strong Chapman singularity in (5.6a,b) and used a multi-regioned technique, based on that developed by Smith (1972, 1974), Daniels (1974, 1976) and others, to deal with the singular behaviour. Thus the solution for Ψ was expressed in the form

$$\Psi = \bar{\xi}\bar{\Psi}_M(\bar{\xi}, \zeta) \qquad (5.8a)$$

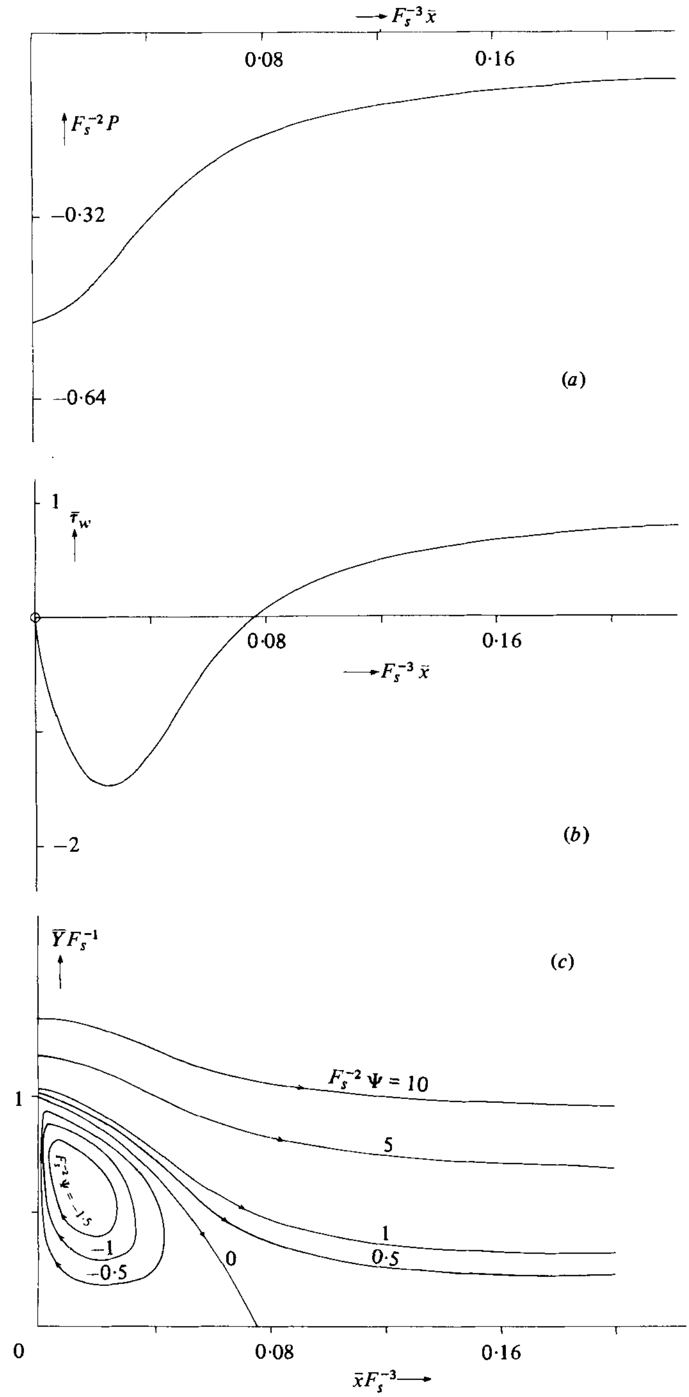

FIGURE 7. The numerical solution for the last phase of the motion (see §5): (*a*) pressure, (*b*) scaled skin friction, (*c*) streamlines.

in a middle domain, where $\bar{\xi} = \bar{x}^{\frac{1}{2}}$ and $\zeta = O(1)$, whereas outside, in an upper and a lower domain, we set

$$\Psi = \Psi_0(\bar{\xi}, \bar{Y}). \tag{5.8b}$$

Then the flow equations in each domain were integrated forward iteratively in uniform steps of $\bar{\xi}$, thus coping with the singularity (5.6a, b). The initial profile f_c was taken as that with $\lambda A_{\frac{1}{4}} = 0$ in (5.6d). At each downstream step the lower and upper domain solutions were required to satisfy the wall condition (5.5d) and the outer constraint (5.5e) respectively while at the common boundary between the middle and the upper or lower domain continuity of Ψ and its first two derivatives with respect to $\bar{Y}$ was ensured. Second-order-accurate differencing with uniform steps in ζ, $\bar{Y}$ as appropriate was used on the first-order differential equations, for Ψ_M, Ψ_0, their first two derivatives with respect to ζ, $\bar{Y}$ respectively, and for P, stemming from (5.3) with (5.9a, b). At $\bar{x} = 1$ a switch to a unified treatment in uniform steps of $\bar{x}$, $\bar{Y}$ could then be safely made. Other details of the procedure follow the patterns set out by Smith (1972, 1974) and Daniels (1974, 1976). The calculation needed to be done only for one value of F_s since $F_s^{-2}\Psi$, $F_s^{-1}U$, $F_s^{-2}P$ in (6.3), (5.5a–e) depend only on $F_s^{-3}\bar{x}$, $F_s^{-1}\bar{Y}$, and we chose the value $F_s = 5$ for convenience in the multi-region approach. Checks were carried out on the effects of the many step sizes involved, as well as the positioning of the outer extremities of the middle and upper domains, and they proved satisfactory. So far results have been obtained with the Reyhner & Flügge-Lotz (1968) approximation invoked whenever backflow occurred, although it is debatable whether or not this is as adequate an approximation as that inherent in the technique of first-order differencing often applied to separated flows, and certainly in the present instance it removes the possibility of a correct description of the detailed behaviour of the solution near $\bar{x} = 0+$. However, as far as the overall properties of the solution are concerned, the proper treatment of the initial Chapman singularity as above is probably as crucial as the influence of the reversed flow especially as the latter turns out to be relatively slow almost throughout. In other problems comparisons with more accurate methods are generally favourable (Williams 1975; Dijkstra 1978) although an improvement of the present calculations, which could presumably be achieved by use of sweeping methods like those of Williams (1975) and Daniels (1979), is clearly desirable. Figure 7 shows the calculated variation with $\bar{x}$ of the pressure P and the skin friction $\bar{\tau}_w \equiv \partial U(\bar{x}, 0)/\partial \bar{Y}$. A notable feature is the absence of any change in sign of the skin friction close to the start of the present stage. This suggests that, just as in Jenson's (1975) calculation of the supersonic flow behind a small backward-facing step, the secondary eddy mentioned earlier is probably only a weak one in practice, a feature not uncommon in multiple eddy formations. Again, the calculated pressure increases in the $\bar{x}$ direction throughout the flow so that according to (5.7c) the influence of the initially reversed flow is very small. The adverse pressure gradient forces the initially separated motion to reattach, in a regular fashion, at a station $\bar{x} = \bar{x}_R \doteqdot 0{\cdot}076F_s^3$, where $\bar{\tau}_w = 0$, and thereafter the pressure and skin friction tend uniformly to their limiting values of zero and unity respectively with increasing distance downstream. The corresponding streamline pattern is shown in figure 7 also. Further refinement of the numerical approach may slightly alter the quantitative properties of the results predicted here but, apart from the secondary eddy mentioned earlier, is hardly likely to disturb their qualitative features and the existence of a solution to the current final stage of the motion is believed to be virtually certain anyway.

6. Further comments

To us there would seem to be three main aspects of the present work worth further comment. The first is the removal of Goldstein's singularity, occurring in the classical boundary-layer approach to separation (§ 2), in the asymptotic description for large h of the deceptively simple-looking flow problem (2.1 a–e). The second aspect concerns the fitting of the present work into broader contexts for internal and external flows at high Reynolds numbers, especially with regard to the nature of separation. Thirdly, there is the matter of the global properties of the solution of § 2–5, particularly as far as the recirculatory eddy or eddies and the long scale reattachment far beyond separation are concerned. Since the Goldstein singularity, or the question of whether its occurrence is welcome or not, is so fundamental to the understanding of high-Reynolds-number flows, however, its removal here and the many corresponding repercussions deserve most comment, initially at least.

The removal of Goldstein's singularity at separation is effected using a sequence of local double structures (§§ 3–4). The first length scale induced (§ 3.1) is fixed primarily by the higher-order properties of the boundary layer, where eigensolutions are forced during the approach to separation (see the end of § 2). On that length scale the singularity is merely given an effective origin shift and a shorter scale variation then sets in (§ 3.2). On the shorter scale the interaction between the pressure disturbance and the displacement becomes just enough to remove the Goldstein singularity and the flow passes through separation in a regular and linear fashion. Only a mild overall pressure rise is produced in the progress through separation and the increase in the wall layer displacement is correspondingly minute. It is perhaps surprising that the interaction has the effect of gradually *reducing* the adverse pressure gradient as the flow passes through separation but this seems to be in line with the idea that the incoming boundary-layer motion is already firmly on its way towards separating and so the originally strong adverse pressure gradient does not need to be enhanced to preserve that trend of the motion. Instead the nature of the pressure–displacement relation is unusual in the sense that the ever-increasing displacement produced by the separation process causes the gradient of the associated pressure disturbance to become increasingly favourable. This interaction is mutually reinforcing and strengthens dramatically beyond separation to produce a second singularity and, thereby, a still shorter length scale for the flow. The new properties on this scale (§ 4) reflect the need for the initial strongly adverse pressure gradient to be reduced still further, since separation has already been accomplished and the next requirement is mainly for the reversed flow velocities to be diminished significantly as the bulk of the boundary layer starts its breakaway from the wall. The character of the second singularity and the resultant new double structure required to remove it exactly fit the above need. For the flow development becomes nonlinear and as a result the incoming adverse pressure gradient starts to suffer a substantial reduction in size. Accordingly the displacement increases, because of the unusual pressure–displacement relation. Indeed, eventually downstream, the adverse pressure gradient tends to zero, the pressure approaches a plateau level and the displacement must then grow indefinitely, giving the breakaway of the bulk of the boundary layer (§ 4). The subsequent flow development is then governed completely by extended free-streamline theory at leading order until at a vast distance downstream the motion enters its final stage (§ 5) where the reattachment, and the recovery to a uniform shear form, take place under the action of viscosity. Comparisons between the overall structure proposed in this paper for the solution of (2.1 a–e) when h is asymptotically large and the calculations for increasing values of h by Smith (1976a) are generally encouraging; see also figure 8.†

Despite the complications it seems clear that the physically sensible removal of Goldstein's singularity hinges on the local pressure–displacement law. For the law is the main difference between our situation, where a meaningful flow development is induced during the process of removal (§§ 3–5), and Stewartson's (1970a), where a removal is found to produce no sensible flow development. There are other differences of course. For instance Stewartson showed that a three-tiered structure is set up by means of an interaction with the mainstream motion present in his case, whereas our case is virtually divorced from any mainstream effects because of the smallness of the length scales involved (see also below) and so is controlled by a sequence of two-tiered structures during separation and breakaway. However, even that difference manifests itself in the respective pressure–displacement laws governing the viscous flow near the wall. Mathematically the significant difference between our small-scale law(s) and most larger-scale law(s) holding under a mainstream is very simple: there is a change of sign [compare (3.4b), (3.12), (4.4) with the laws of supersonic, hypersonic, transonic or channel flow, for example]. The theoretical and physical repercussions are quite profound nonetheless. In Stewartson's (1970a) situation the local law allows, first, eigensolutions, i.e. free interactions, to develop upstream and, second, a physically unrealistic singularity to occur during the removal process, at least in supersonic flow where the law is Ackeret's. Our local law allows neither of these occurrences. It insists that the pressure and the negative displacement increases or decrease together in view of the nonlinear balance (2.4) which turns out to hold throughout our flow provided $F(x)$ is replaced by the appropriate unknown displacement during and beyond separation: see (3.4b), (3.12), (4.4), (5.2b) and Smith (1979a), Smith & Duck (1980). Thus at any station an extra increase, say, in pressure would tend to decrease the skin friction and push the wall layer slightly away from the wall, through the viscous response; but then the implied increase in displacement contradicts the pressure–displacement law, so that the interaction is damped and cannot develop. Consequently uniqueness of the solution seems to be favoured in our case. Given the physical sense of the original governing equations the ensuing unique flow development is then almost bound to be realistic. Hence beyond our separation the singular

† In addition quantitative comparisons are given in figure 9(a)–(e) for the obstacle

$$F(x) = x \exp(-x^2/32) \quad \text{in} \quad x \geqslant 0, \quad F(x) = 0 \quad \text{in} \quad x \leqslant 0,$$

chosen by Smith (1976a) and for which $x_{\max} = 4$. First, figures 9(a, b) compare his results for the pressure and skin friction with the theory of § 2 up to separation. We calculated the solution of the classical boundary-layer problem (2.6a–e) using a numerical scheme similar to those used before (§§ 4, 5) and integrated up to the onset of the Goldstein singularity at separation, which was found to occur at approximately $x_s = 4{\cdot}78$, giving $F_s = 2{\cdot}341$. The calculated skin friction and boundary-layer displacement are given in figure 9(b). The agreement between the present theory and Smith's results is fairly encouraging, especially when the higher-order effect (2.7) in the pressure expansion of (2.2) is allowed for. Second, figures 9(c, d) compare his results with the theory of § 5 on the suggested long scale of $O(h^3)$ beyond separation. Again the overall trend is fairly supportive of the present theory. Lastly, figure 9(e) presents the dependence of the separation and reattachment positions on h, according to Smith's calculations and our theory, which are found to be in reasonable agreement.

behaviour has the displacement increasing sharply, therefore both the local pressure and skin friction fall rapidly and the increasingly less adverse pressure gradient is physically sensible when considered in the light of the breakaway of the boundary layer downstream, as noted before. Likewise the remainder of the flow development retains physical reality (§ 5). Returning to the larger-scale situation of Stewartson (1970*a*), we can see that although the pressure–displacement law there leads to an unrealistic singularity the law nevertheless hints at the ultimate resolution of all the difficulties inherent in the classical approach there. For the upstream interactions which the law allows to take place upstream in Stewartson's (1970*a*) modification of the Goldstein singularity have exactly the same qualitative balance as that which sets off regular separation in the alternative, self-consistent, theory of triple-deck structures, in supersonic flow. Any pressure rise causes the displacement also to rise both through the viscous response of the wall layer and the pressure–displacement law and so the interaction is self-supporting. This free interaction can develop in a well-attached boundary layer, to force a sudden separation, just as it can develop in the nearly separated boundary layer produced from a slow classical approach; but fortunately the subsequent failure in the classical approach is not reproduced in the triple-deck approach. Instead the latter approach provides the required complete theoretical account of regular separation and breakaway, in supersonic, subsonic and many other large-scale flows. All in all, the different natures of the pressure–displacement relations completely dominate matters. It is tempting to conclude from the above that depending on the local pressure–displacement relation a proper account of separation can always be gathered either from the classical approach or, failing that, from the free-interaction approach, although such a conclusion is perhaps too simple to be true given all the possible complicated features which can arise in high-Reynolds-number flows with separation.

We hasten to add here that in our opinion the realistic removal of Goldstein's singularity at separation in the present work almost certainly does not disturb the powerful standpoint, developed in recent years, that *in general* the Goldstein singularity cannot be removed in any physically sensible way; that consequently its occurrence in a classical boundary-layer theory usually means in effect that such a theory is incorrect; and that the alternative theory, based instead on the local interactive description of separation (by means of triple decks, double decks and other structures as appropriate: see the summaries by Stewartson 1974*a, b*, 1980; Messiter 1979; Smith 1979*c*), then provides the proper basis for (*inter alia*) the asymptotic description of separating flows. Rather, the present work simply raises the possibility that in certain flow situations the occurrence of the singularity may not be catastrophic for the theory used in the approach to separation. To attempt to name many candidates for such certain flow situations would seem too risky at the moment and indeed the ultimate list, of the really distinct flow situations which do admit sensible removal of the Goldstein singularity, is probably a very short one. All we can claim, with some conviction, is that the present flow situation is on that list. It might be anticipated also that the three-dimensional or unsteady extensions of the present flow situation studied by Smith (1976*c*, 1980) and Sykes (1980) belong to the list. Moreover, although the possibility of others may well be worth some further exploration, the following discussion of our work in terms of the internal and external motions that produced it (see §§ 1, 2), it is hoped, will put the matter accurately into a broader context.

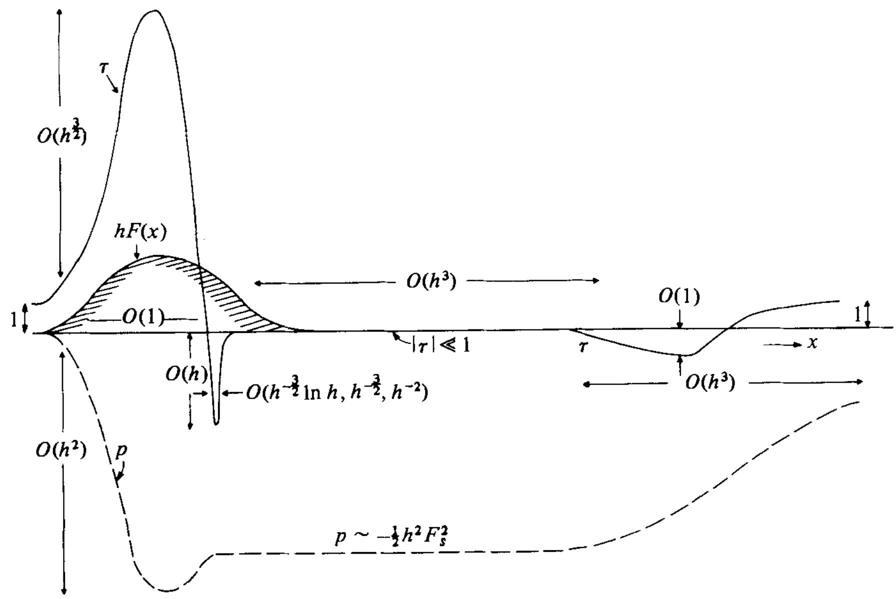

FIGURE 8. Diagram, not to scale, summarizing the behaviour of the pressure (dashed curve) p and skin friction (solid curve) $\tau \equiv (\partial u/\partial y)\,(x,\,0)$ predicted throughout the flow field when h is large. Here §2 describes the properties for $x < x_s$, §3–4 for $|x-x_s| \ll 1$ and §5 for $x > x_s$, with τ being given by $h^{\frac{3}{2}}\tau_0$ in §2, $h^{\frac{3}{4}}(\ln h)^{\frac{1}{2}}\mu\hat{\alpha}_1$ in §3.1, $h^{\frac{3}{4}}\mu\bar{\alpha}_1$ in §3.2, $h\hat{\tau}_w$ in §4 and by $\bar{\tau}_w$ in §5. The predictions are compared with Smith's (1976*a*) results in figure 9 below.

Concerning external motions, whether supersonic or subsonic, with a global Reynolds number Re, the problem (2.1*a–e*) arises when an attached boundary layer of thickness $O(Re^{-\frac{1}{2}})$ and typical streamwise scale $O(1)$ encounters a shallow obstacle whose length and height scales are $l\,Re^{-\frac{1}{2}}$ and $l^{\frac{1}{3}}\,Re^{-\frac{2}{3}}h$ respectively, but with

$$Re^{-\frac{1}{4}} \ll l \ll Re^{\frac{1}{8}}.$$

The restriction here on l means that the obstacle's dimensions are much less than those of the triple-deck structure (Smith 1973; Smith *et al.* 1981), accounting for the lack of an unknown displacement effect in (2.1*d*). A summary of the flow properties produced when $h = O(1)$ or smaller is given by Smith *et al.* (1981) but our interest lies in the more significant limit $h \to \infty$ with $l = O(1)$ where the obstacle steepens and produces separation. There the eddy length increases like h^3, from §5. Accordingly when the obstacle height $l^{\frac{1}{3}}\,Re^{-\frac{2}{3}}h$ rises to $O(Re^{-\frac{5}{8}})$, the triple-deck height, so that h becomes $O(l^{-\frac{1}{3}}\,Re^{\frac{1}{24}})$ formally, the eddy length, $l\,Re^{-\frac{1}{2}}h^3$, becomes $O(Re^{-\frac{3}{8}})$ for any given obstacle length scale l. The inference we draw from this is that as we increase the height of any obstacle whose length lies between $O(Re^{-\frac{3}{4}})$ and $O(Re^{-\frac{3}{8}})$ the grossly separated flow structure of §§2–5 above continues to hold [once the stage $h = O(1)$ identified by Smith (1973), Smith *et al.* (1981) has been passed through] until the obstacle height reaches the triple-deck size $O(Re^{-\frac{5}{8}})$. Then, however, the triple-deck interaction between the induced pressure and the displacement and the influence of the uniform flow just outside the boundary layer must come into play in the control of the long wake motion, with the obstacle appearing as a normal flat plate, or fence, over the wake length scale (figure 8). The wake properties are then bound to change from those of §5. Eventually separation upstream of the effective fence may also be anticipated when the obstacle height lies within the triple-deck regime. The flow properties for still higher obstacles depend on the nonlinear triple-deck solution. It is interesting that within the triple-deck stage although the obstacle acts effectively as a normal fence the separation streamline emerges, from the more local length scale of the obstacle itself, at a height initially less than the height of the fence (since $F_s < F_{\max}$ in § 2). More important, however, is the conclusion that eventually the triple-deck balance does return to take control of the majority of the flowfield, for steeper obstacles, and it seems definite that thereafter, as we move on towards the flow induced by a severe or bluff obstacle, the interactive triple-deck descriptions for separation and reattachment provide all the necessary clues to the flow structure, as opposed to the classical description which is limited to the much smaller disturbances discussed just previously.

There seems less of a limitation on the applicability of the classical approach of § 2 and the removal of Goldstein's singularity in §§ 3–4 when the other context of internal motions through tubes is considered. Here with a global Reynolds number R, based on the pipe radius or channel width and, say, the maximum speed of the oncoming flow, which could be Poiseuillean, the problem (2.1 *a–e*) is encountered when a symmetric 'fine constriction' of non-dimensional length $O(1)$ and height $O(hR^{-\frac{1}{3}})$ is made in the tube (Smith 1976 *a, b*). Let us examine the implications for increasingly steep symmetric constrictions ($h\uparrow$). Smith (1978) showed that a new structure takes over when the constriction height reaches $O(R^{-\frac{1}{6}})$ ('moderate constriction') as the flow ahead of the constriction then has to adjust nonlinearly and can separate. The upstream adjustment depends on the global perturbation solution in the core of the flow, however, and that in turn depends on the global shape of the eddy downstream of the constriction. This eddy has length $O(h^3) = O(R^{\frac{1}{2}})$ since formally h rises to $O(R^{\frac{1}{6}})$ for moderate constriction; see also Smith (1978). On the other hand although the eddy boundary is still a straight line parallel to the tube axis on the $O(1)$ length scale, it emanates from the separation position on the constriction (as in § 5) which is still determined from the classical boundary layer on the constriction. Therefore the core flow perturbation and the upstream adjustment can be fixed only after the position of separation, with its removable Goldstein singularity, has been determined from the nonlinear classical boundary-layer solution. It is fortunate that the flow adjustment upstream of the constriction does not influence the classical layer on the constriction, incidentally. The option of a removable Goldstein singularity rather than a Sychev (1972)–Smith (1977 *a*) triple deck at separation was raised by Smith (1978) and seems certain in view of §§ 2–5 above, although the latter paper avoided the difficulty by imposing restrictions on the hump shape so that other flow features could be studied instead. In contrast, when the constriction becomes a 'severe' one of height $O(1)$ comparable with the tube's cross-sectional dimensions, the core flow is so strongly affected (Smith 1979 *a*) that in the neighbourhood of separation the main flow properties are those characteristic of an external motion. Thus the Sychev–Smith description of separation seems required then, since the option of a sensibly removable Goldstein singularity is ruled out by the relevance of Stewartson's (1970 *a*) analysis for the effectively external local flow subjected to a uniform mainstream in place of our analysis (§§ 2–5) where no uniform mainstream can make itself felt. Between the severe and the moderate cases of constriction, therefore, there may presumably be a gradual or a sudden transition from the classically produced separation holding for moderate

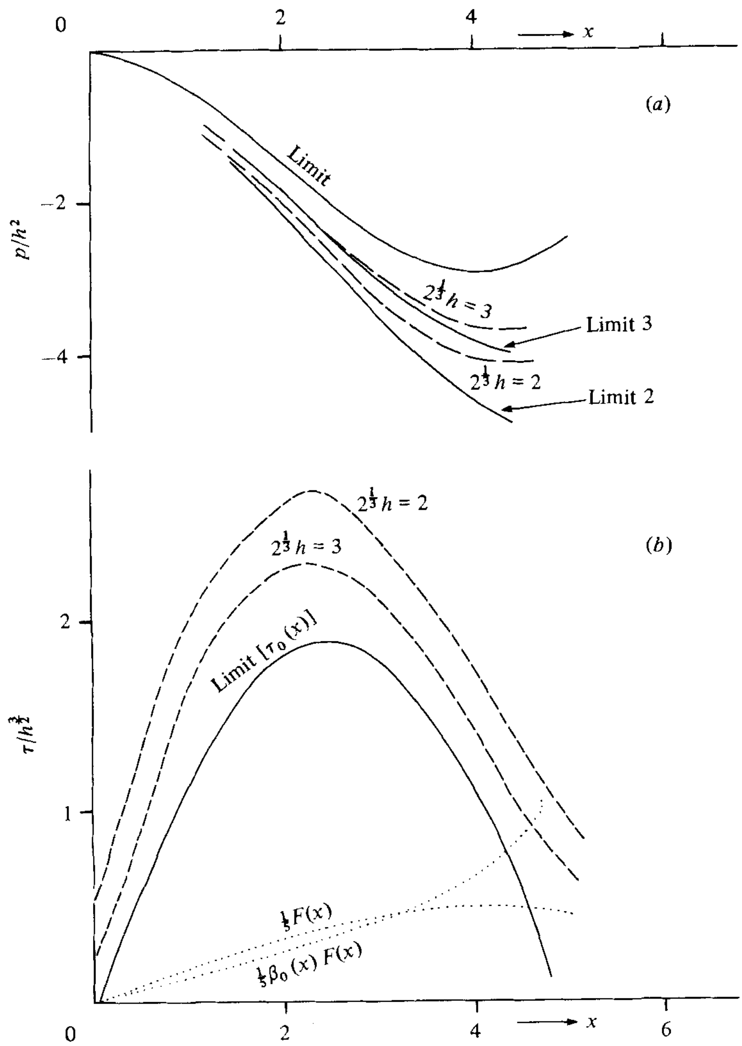

FIGURE 9 (*a*, *b*). For the caption see page 36.

constriction to the interactively produced type holding for severe constriction. This would seem well worth studying because *inter alia* it would explain the overall forward movement required of the separation point on the constriction as we progress from moderate constriction, where (§§ 2–5) separation occurs at an $O(1)$ distance beyond the maximum constriction point, to the onset of severe constriction where the flow separates near the maximum constriction point in general (Smith 1979*a*). The above discussion can be modified, for the application to nonsymmetric internal flows where Smith's (1977*b*) and Smith & Duck's (1980) studies propose the flow structure for severe constriction, to non-Poiseuillean oncoming flows and to other forms of boundary conditions.

Finally, addressing the structure of the large eddy and long scale reattachment discussed in § 5, we believe that the self-consistency demonstrated there firmly supports the view that extended free-streamline theory gives the correct inviscid limiting

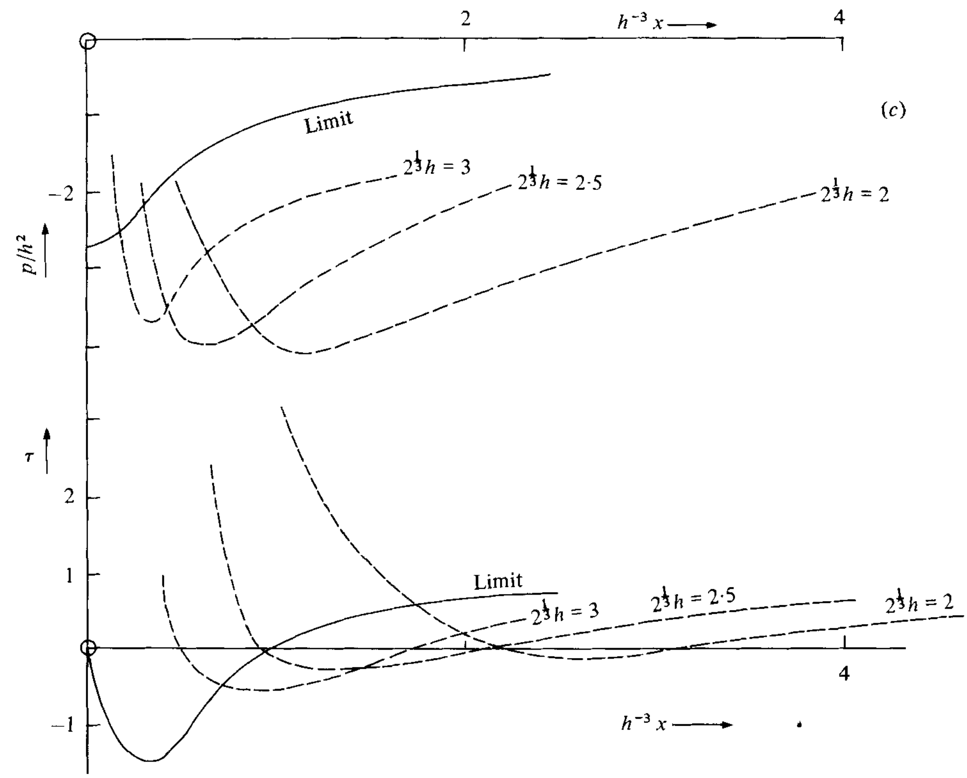

FIGURE 9 (c d). For the caption see next page.

solution (*a*) certainly for grossly separated internal flows and (*b*) quite possibly for most grossly separated external flows. With regard to (*a*), the eddy and reattachment structure of flow through constricted tubes stays very similar to the structure set out in § 5, beyond the constriction, even when the constriction is much bigger (Smith 1978, 1979*a*). The downstream reattachment continues to be strongly influenced by the viscous forces produced over the vast length scale beyond separation and that allows the free-streamline approach ahead of the reattachment phase to remain intact since the action of viscosity further downstream is able to prevent the return of any significant back flow. So the major extra complication [see also the previous paragraph] produced by bigger constrictions concerns only the flow ahead of the constriction (Smith 1978, 1979*a*) where in general a substantial separation also occurs eventually. There again, however, the back flow produced at the corresponding reattachment, where the shear layer emanating from the upstream separation meets the front of the constriction, is found to be small (Smith 1979*a*; Smith & Duck 1980) and so again does not disturb the properties of the extended free-streamline approach, allied properly to the appropriate viscous–inviscid interaction at separation, upstream. Comparisons of the theory with numerical solutions of the Navier–Stokes equations (Dennis & Smith 1980) up to Reynolds numbers of 2000 and with experimental findings (Smith 1979*a*) add further weight to the present view. With regard to (*b*) above we can claim that for the obstacles (severely disturbing an external boundary layer) for which (2.1*a*–*e*) holds with h large the extended free-streamline approach is self-consistent

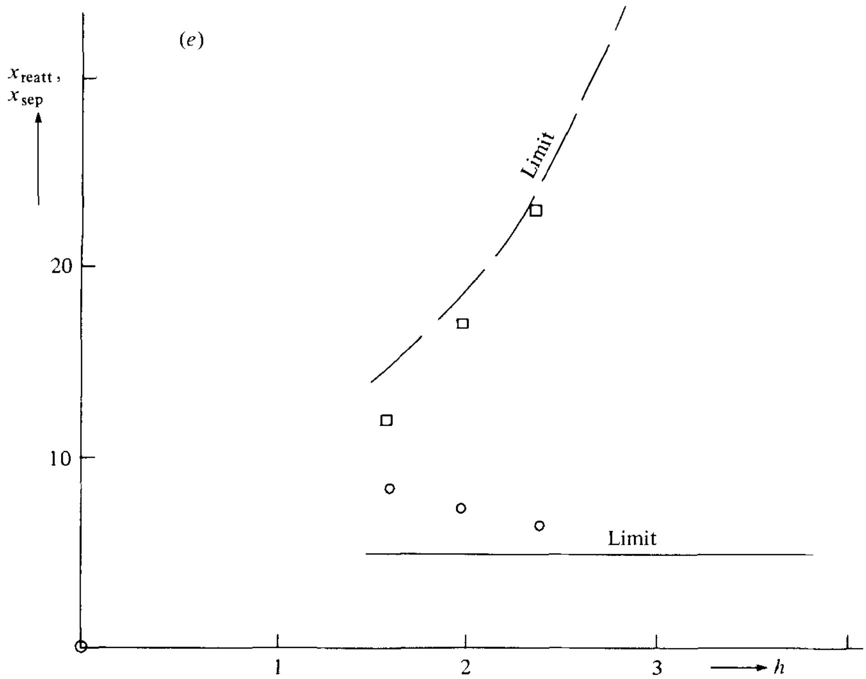

FIGURE 9. Comparisons between the present theory and Smith's (1976*a*) results (his figure 5) for the obstacle $F(x) = xe^{-x^2/32}$ when $x \geqslant 0$, $F(x) = 0$ when $x \leqslant 0$.

(*a*) Pressure $h^{-2}p$ versus x: – – –, from Smith (1976*a*) at values of h shown; ——, from §2. 'Limit' gives the prediction (2.2) with (2.4) only, while 'Limit 2' and 'Limit 3' include the higher-order effect (2.7), for $2^{\frac{1}{3}}h = 2, 3$ respectively, where the displacement $\beta_0(x)$ in (2.7) follows from (*b*) below.

(*b*) Skin friction $h^{-\frac{3}{2}}\tau$ versus x: – – –, as in (*a*); ——, τ_0 from the solution of the classical boundary layer (2.6*a*–*e*). Also shown (... ...) for reference are $F(x)$ [the obstacle shape, or external velocity driving the boundary layer, from (2.6*c*)] and $\beta_0(x)$ $F(x)$ which fixes the displacement $\beta_0(x)$ of (2.6*e*). The Goldstein singularity as $x \to x_s - \doteqdot 4{\cdot}78-$ seems fairly evident in the behaviour of $\tau_0(x)$ and $\beta_0(x)$.

(*c*) Pressure $h^{-2}p$ versus $h^{-3}x$: – – –, as in (*a*); ——, from §5 (figure 7*a*). Here $F_s \doteqdot 2{\cdot}341$.

(*d*) Skin friction τ versus $h^{-3}x$: – – –, as in (*a*); ——, from §5 (figure 7*b*).

(*e*) Positions of separation (x_{sep}) and reattachment (x_{reatt}) as functions of h: ○, □ give x_{sep}, x_{reatt} respectively from Smith (1976*a*), while for $h \gg 1$ the solid line is the prediction $x_{sep} \sim x_s$ from the solution of (2.6*a*–*e*) and the dashed line the prediction $x_{reatt} \sim h^3\bar{x}_R$ from §5 (see also figure 7*b*). An origin shift of 10 has been applied in the latter prediction.

again due to the action of viscosity during reattachment. On the other hand an increase of the obstacle height, or indeed its length, to the triple-deck size provokes a strong interaction with the mainsteam and thus eventually a probable change in the character of the reattachment process (see the studies by Messiter, Hough & Feo 1973; Daniels 1979) as well as in the separation process as described previously. The weakness of the external mainstream effect in the present work [i.e. in (2.1*a*–*e*)], before that triple-deck stage is reached, is responsible for the likeness of the external flow properties then to internal flow properties. Whether or not the extended free-streamline approach continues to hold for still bigger obstacles remains to be settled fully, quite possibly by means of accurate numerical solutions of the triple-deck problem coupled with more asymptotic understanding of eddies and reattachment. There is strong evidence, however (Burggraf 1975; Smith 1979*b*), in favour of the opinion that the interactive or extended free-streamline theory is also the correct limit for much bigger disturbances in external flows.

REFERENCES

BROWN, S. N. & STEWARTSON, K. 1969 *Ann. Rev. Fluid Mech.* **1**, 45.

BURGGRAF, O. R. 1975 *AGARD Conf. Proc.* 168.

CATHERALL, D. & MANGLER, K. W. 1966 *J. Fluid Mech.* **26**, 163.

DANIELS, P. G. 1974 *J. Fluid Mech.* **63**, 641.

DANIELS, P. G. 1976 *J. Inst. Maths Applics* **17**, 367.

DANIELS, P. G. 1979 *J. Fluid Mech.* **90**, 289.

DIJKSTRA, D. 1978 *Proc. 6th Int. Conf. Num. Meth. Fluid Dyn., Tbilisi, U.S.S.R.*

DENNIS, S. C. R. & SMITH, F. T. 1980 *Proc. Roy. Soc.* A **372**, 393.

GOLDSTEIN, S. 1948 *Quart. J. Mech. Appl. Math.* **1**, 43.

JENSON, R. 1975 Ph.D. thesis, Ohio State University.

KIRCHHOFF, G. 1869 *J. reine angew. Math.* **70**, 289.

MESSITER, A. F. 1970 *SIAM J. Appl. Math.* **18**, 241.

MESSITER, A. F. 1975 *AGARD Conf. Proc.* p. 168.

MESSITER, A. F. 1979 *Proc. 8th U.S. Nat. Cong. of Appl. Mech.*, 1978, *Los Angeles, U.S.A.*

MESSITER, A. F., HOUGH, G. R. & FEO, A. 1973 *J. Fluid Mech.* **60**, 605.

PROUDMAN, I. 1960 *J. Fluid Mech.* **9**, 593.

REYHNER, T. A. & FLÜGGE-LOTZ, I. 1968 *Int. J. Nonlinear Mech.* **3**, 173.

SMITH, F. T. 1972 D.Phil. thesis, Oxford University.

SMITH, F. T. 1973 *J. Fluid Mech.* **57**, 803.

SMITH, F. T. 1974 *J. Inst. Maths Applics* **13**, 127.

SMITH, F. T. 1976*a*, *b* *Quart. J. Mech. Appl. Math.* **29**, 343; **29**, 365.

SMITH, F. T. 1976*c* *Mathematika* **23**, 62.

SMITH, F. T. 1977*a* *Proc. Roy. Soc.* A **356**, 443.

SMITH, F. T. 1977*b* *J. Fluid Mech.* **79**, 631.

SMITH, F. T. 1978 *J. Inst. Maths Applics* **21**, 145.

SMITH, F. T. 1979*a* *J. Fluid Mech.* **90**, 725.

SMITH, F. T. 1979*b* *J. Fluid Mech.* **92**, 171.

SMITH, F. T. 1979*c* Lecture course, *Asymptotic Methods in Appl. Mech., C.I.S.M., Udine, Italy.* (Also to appear in *J. Inst. Math. Applic.*, 1981.)

SMITH, F. T. 1980 *J. Fluid Mech.* **99**, 185.

SMITH, F. T., BRIGHTON, P. W. M., JACKSON, P. S. & HUNT, J. C. R. 1981 *J. Fluid Mech.* (to appear).

SMITH, F. T. & DUCK, P. W. 1977 *Quart. J. Mech. Appl. Math.* **30**, 143.

SMITH, F. T. & DUCK, P. W. 1980 *J. Fluid Mech.* **90**, 727.

SMITH, F. T. & STEWARTSON, K. 1973 *J. Fluid Mech.* **58**, 143.

STEWARTSON, K. 1970*a* *J. Fluid Mech.* **44**, 347.

STEWARTSON, K. 1970*b* *Mathematika* **16**, 106.

STEWARTSON, K. 1974*a* *Adv. Appl. Mech.* **14**, 145.

STEWARTSON, K. 1974*b* *J. Fluid Mech.* **62**, 289.

STEWARTSON, K. 1980 *The von Neumann Lecture, Washington D.C., U.S.A.*

STEWARTSON, K. & WILLIAMS, P. G. 1969 *Proc. Roy. Soc.* A **312**, 181.

STEWARTSON, K. & WILLIAMS, P. G. 1973 *Mathematika* **20**, 98.

SYCHEV, V. V. 1967 *Rep. 8th. Symp. Recent Problems in Mech. Liquids & Gases, Tarda, Poland.*

SYCHEV, V. V. 1972 *Izv. Akad. Nauk S.S.S.R. Meth. Zhidk. i Gaza* **3**, 47.

SYKES, R. I. 1980 *Proc. Roy. Soc.* A **373**, 311.

WILLIAMS, P. G. 1975 *Proc. 4th Int. Conf. Num. Meths. in Fluid Dyn., Boulder* 1974. Springer.